Finite and boundary elements for interface problems in nonlinear elasticity: adaptivity and hp–stabilization

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Outline

- Nonlinear contact problems: examples, pictures, set–up
- FE–BE coupling, error estimates, adaptivity
- Numerical examples
- Stabilized mixed high–order boundary elements

Two references:

HG, M. Maischak, E. Schrohe, E. P. Stephan, *Adaptive FE–BE coupling for strongly nonlinear transmission problems with Coulomb friction*, Numer. Math. (2011), 26 pages.

L. Banz, HG, A. Issaoui, E. P. Stephan, *Stabilized mixed hp-BEM for frictional contact problems in linear elasticity*, Numer. Math. (2016), 47 pages.

Laplace equation as energy minimization

$\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain, $f \in H^{-1}(\Omega)$.

Recall:

$u \in H^1(\Omega)$ solves the Dirichlet problem $-\Delta v = f, \;\; v|_{\partial\Omega} = 0$
$\iff u$ minimizes $E(v) = \frac{1}{2}\int_\Omega (\nabla v)^2 - \langle f, v\rangle$ over $H_0^1(\Omega)$

inhomogeneous/anisotropic energy:

$E(u) = \frac{1}{2}\int_\Omega (\nabla u(x))^t D(x) \nabla u(x) - \langle f, u\rangle$, $D = D^t > 0 \rightsquigarrow -\mathrm{div}\,(D\nabla u) = f$

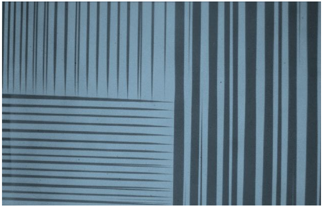
Examples of nonlinear operators

$$E(u) = \frac{1}{2}\int_\Omega (\nabla u(x))^t D(x) \nabla u(x) - \langle f, u\rangle \rightsquigarrow -\text{div}\,(D\nabla u) = f$$

p-Laplace: $E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \langle f, u\rangle \rightsquigarrow -\text{div}\,(|\nabla u|^{p-2}\nabla u) = f$, $p \in (1, \infty)$

- models non-Newtonian fluids, Hencky materials in elasticity
- $p < 2$ hair gel, glaciers, $p > 2$ thick emulsion of sand and water

phase transitions / bistable materials: double–well potential

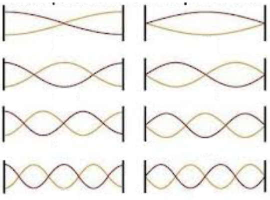


(from S. Müller)

$$E(u) = \int |\nabla u - (1,0)|^2 \, |\nabla u - (0,1)|^2 - \langle f, u\rangle$$

Boundary conditions

Neumann and Dirichlet:



Contact: Signorini (= nonpenetration, $\perp$ wall) and friction ($\|$ wall)

(Our results are new even without these.)

Nonlinear transmission problems with friction

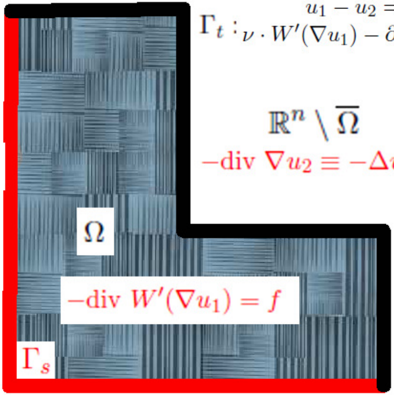
<u>transmission</u>

$u_1 - u_2 = u_0$

$\Gamma_t : \nu \cdot W'(\nabla u_1) - \partial_\nu u_2 = t_0$

$\mathbb{R}^n \setminus \overline{\Omega}$

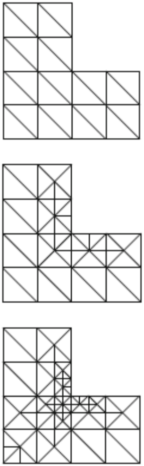
$-\mathrm{div}\ \nabla u_2 \equiv -\Delta u_2 = 0$



$\nu \cdot W'(\nabla u_1) - \partial_\nu u_2 = t_0, \quad |\nu \cdot W'(\nabla u_1)| \leq g$

$\nu \cdot W'(\nabla u_1)(u_0 + u_2 - u_1) + g|(u_0 + u_2 - u_1)| = 0$

<u>contact</u>



How bad can W' be? p-Laplace, double-well

Adaptive: Refine where numerical error is large!

FE/BE approximation

Equivalent formulation $W \leftrightarrow W'$:

$$\begin{aligned} E(u_1, u_2) &= \int_\Omega W(\nabla u_1) + \frac{1}{2}\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u_2|^2 - \int_\Omega f u_1 - \langle t_0, u_2|_{\partial\Omega}\rangle \\ &\quad + j((u_2 - u_1 + u_0)|_{\Gamma_s}) \;\rightarrow\; \min_{\mathcal{C}} \end{aligned}$$

$$\mathcal{C} = \{(u_1, u_2) \text{ of finite energy} : (u_1 - u_2)|_{\Gamma_t} = u_0 \;,\; u_2 \in \mathcal{L}_2\},$$

$$\mathcal{L}_2 \simeq \left\{ w : \Delta w = 0 \;\text{ in } \mathbb{R}^n \setminus \overline{\Omega}, \, w \to 0 \text{ at } \infty \right\}.$$

$$j(v) = \int_{\Gamma_s} g|v|$$

Friction: $E \notin C^1$, because $j \notin C^1$.
Other contact conditions: $E \notin C^1$ or $\min_{\mathcal{C}}$ ($\mathcal{C}$ closed convex).

Green's formula $\quad \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u_2|^2 = \int_{\partial\Omega} (-\partial_\nu u_2) u_2 =: \int_{\partial\Omega} (S u_2|_{\partial\Omega}) u_2|_{\partial\Omega}$.

S Dirichlet–Neumann operator, ΨDO on $\partial\Omega$.

FE/BE approximation

Equivalent formulation $W \leftrightarrow W'$:

$$E(u_1,u_2) \;= \int_\Omega W(\nabla u_1) + \frac{1}{2}\int_{\partial\Omega}(Su_2|_{\partial\Omega})u_2|_{\partial\Omega} - \int_\Omega f u_1 - \langle t_0, u_2|_{\partial\Omega}\rangle + j((u_2 - u_1 + u_0)|_{\Gamma_s}) \;\rightarrow\; \min_{\mathcal{C}}$$

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S Dirichlet–Neumann operator, ΨDO on $\partial\Omega$.

$$u_2 = u_1 - u_0 + v, \;\; \operatorname{supp} v \subset \Gamma_s \;.$$

FE/BE approximation

Equivalent formulation $W \leftrightarrow W'$:

$$E(u,v) = \int_\Omega W(\nabla u) + \frac{1}{2}\int_{\partial\Omega}(S(u|_{\partial\Omega}+v))(u|_{\partial\Omega}+v) - \int_\Omega fu - \langle t_0, v\rangle + j(v) + C \rightarrow \min_{\mathcal{C}}$$

$$\mathcal{C} = \{(u,v) \text{ of finite energy} : \operatorname{supp} v \subset \Gamma_s\},$$

Friction: $E \notin C^1$, because $j \notin C^1$.
Other contact conditions: $E \notin C^1$ or $\min_{\mathcal{C}}$ ($\mathcal{C}$ closed convex).

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S Dirichlet–Neumann operator, ΨDO on $\partial\Omega$.

$$u_2 = u_1 - u_0 + v, \quad \operatorname{supp} v \subset \Gamma_s .$$

FE/BE approximation

Equivalent formulation $W \leftrightarrow W'$:

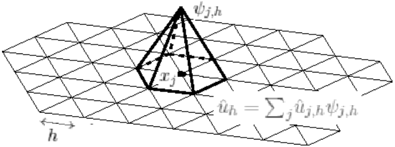
$$E(u,v) \quad = \int_\Omega W(\nabla u) + \frac{1}{2}\int_{\partial\Omega}(S(u|_{\partial\Omega}+v))(u|_{\partial\Omega}+v) - \int_\Omega f - \langle t_0, v\rangle + j(v) + C \;\to\; \min_{\mathcal{C}}$$

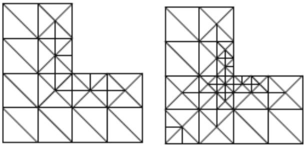
Numerical approximation: **F**inite **E**lements / **B**oundary **E**lements

(((+ convexify if W nonconvex)))

$$E_h(u_h,v_h) = \int_\Omega W(\nabla u_h) + \frac{1}{2}\int_{\partial\Omega}(S_h(u_h|_{\partial\Omega}+v_h))(u_h|_{\partial\Omega}+v_h) + L(u_h,v_h) + j(v_h) \;\to\; \min_{\mathcal{C}_h}$$

$$\mathcal{C}_h = \{(u_h,v_h)\;:\; u_h = \sum_j u_{j,h}\psi_{j,h},\; v_h = \sum_{j'} v_{j,h}\psi_{j',h}|_{\partial\Omega}\}\;.$$





Dirichlet–Neumann operator – Why?

$$\int_{\Omega^c} |\nabla u_2|^2 = \int_{\partial\Omega} (-\partial_\nu u_2|_{\partial\Omega}) u_2|_{\partial\Omega} = \int_{\partial\Omega} (S u_2|_{\partial\Omega}) u_2|_{\partial\Omega}$$

$+$ $\dim \partial\Omega = n - 1$, $\partial\Omega$ bounded

$+$ hp-boundary element methods: exponential convergence

$+$ compression/preconditioning: dense matrices ok

$-$ homogeneous linear equations

$-$ dense matrices: storage, time $\sim (\text{DOF})^2$, need optimization

Dirichlet–Neumann operator / Multi-layer potentials

S elliptic pseudodifferential operator of order 1 on $\partial\Omega$ (if $\partial\Omega \in C^\infty$)

$H^{\frac{1}{2}}(\partial\Omega)$–coercive: $\langle Su|_{\partial\Omega}, u|_{\partial\Omega}\rangle \geq \alpha \|u\|^2_{H^{\frac{1}{2}}(\partial\Omega)}$

Computation via multi-layer potentials: $S = \mathcal{W} + (1-\mathcal{K}')\mathcal{V}^{-1}(1-\mathcal{K})$

$$\mathcal{V}\phi(x) = \int_{\partial\Omega} k(x,y)\phi(y)\, ds_y\ , \qquad k(x,y) = \begin{cases} -\frac{1}{2\pi}\log|x-y|, & n=2 \\ \frac{1}{4\pi}\frac{1}{|x-y|}, & n=3. \end{cases}$$

$\mathcal{W}$, $\mathcal{K}$, $\mathcal{K}'$ similar using normal derivatives of k.

Final set–up

$p \geq 2: \quad X^p = W^{1,p}(\Omega) \times \widetilde{H}^{\frac{1}{2}}(\Gamma_s), \quad \widetilde{H}^{\frac{1}{2}}(\Gamma_s) = \{u \in H^{\frac{1}{2}}(\partial\Omega) : \text{supp}\, u \subset \bar{\Gamma}_s\}$

(VI): Find $(\hat{u}, \hat{v}) \in X^p$ s. t. $\forall (u, v) \in X^p$:

$$\langle W'(\nabla \hat{u}), \nabla(u - \hat{u})\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u - \hat{u})|_{\partial\Omega} + v - \hat{v}\rangle + j(v) - j(\hat{v}) \;\geq\; L(u - \hat{u}, v - \hat{v}) \; .$$

Theorem

(VI) is equivalent to contact problem / energy minimization, unique solution if $W(\nabla u)$ strictly convex.

Final set–up

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$(VI)_h$: Find $(\hat{u}_h, \hat{v}_h) \in X_h^p$ s. t. $\forall (u_h, v_h) \in X_h^p$:

$$\langle W'(\nabla \hat{u}_h), \nabla (u_h - \hat{u}_h)\rangle + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h\rangle$$
$$+ j(v_h) - j(\hat{v}_h) \;\geq\; L(u_h - \hat{u}_h, v_h - \hat{v}_h)\ .$$

Theorem

$(VI)_h$ *is equivalent to discretized energy minimization, unique solution if* $W(\nabla u)$ *strictly convex.*

Final set–up

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Theorem

$(VI)_h$ is equivalent to discretized energy minimization, unique solution if $W(\nabla u)$ strictly convex.

$$p < 2: \quad W^{1,p}(\Omega) \ni u \mapsto \langle Su|_{\partial\Omega}, u|_{\partial\Omega}\rangle \text{ not continuous!}$$

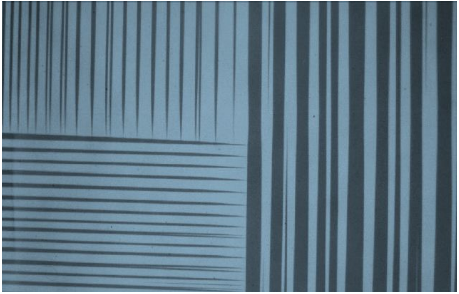
$$X^p = \{(u,v) \in W^{1,p}(\Omega) \times \widetilde{W}^{1-\frac{1}{p},p}(\Gamma_s) : u + v \in H^{\frac{1}{2}}(\partial\Omega)\}$$

Theoretical results

Monotone problems

a) unique solutions $\hat{u}$, $\hat{u}_h$
b) convergence: $\|\hat{u}-\hat{u}_h, \hat{v}-\hat{v}_h\|_{X^p} \to 0$ if $\operatorname{dist}(X_h^p, X^p) \to 0$
c) a posteriori bounds for error in Ω and $\partial\Omega$ can be coupled (adaptivity)

Nonconvex functionals: Microstructure



Double-well (with Signorini contact):

$$\int_\Omega W(\nabla u) = \int_\Omega |\nabla u - (-1,0)|^2 \, |\nabla u - (0,1)|^2$$

No existence / No uniqueness: Young measure solutions
Convexification: stable computation of macroscopic quantities

Theoretical results

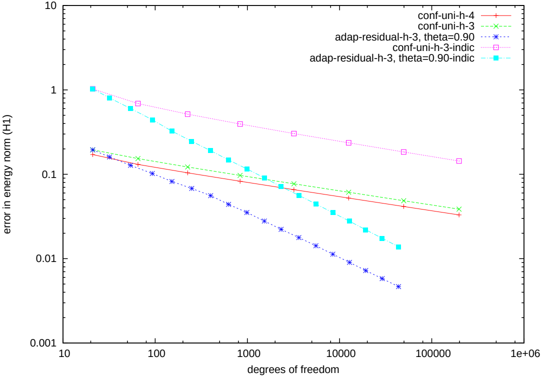
Monotone problems

a) unique solutions $\hat{u}$, $\hat{u}_h$
b) convergence: $\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h\|_{X^p} \to 0$ if $\mathrm{dist}(X_h^p, X^p) \to 0$
c) a posteriori bounds for error in Ω and $\partial\Omega$ can be coupled (adaptivity)

Nonconvex problem: double-well

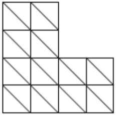
a) solution not unique
b) all solutions have same stress $DW^{**}(\hat{u})$, region of microstructure, boundary value $\hat{u}|_{\partial\Omega} + \hat{v}$, some directional derivatives
c) convergence for these quantities as $\mathrm{dist}(X_h^p, X^p) \to 0$
d) a posteriori bounds for these quantities can be coupled (adaptivity)

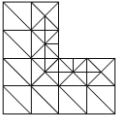
Adaptive algorithm in practice

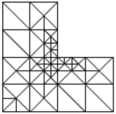


$$W(\nabla u) = (\varepsilon + |\nabla u|)^p,$$

$p = 3$ and $\varepsilon = 10^{-5}$,
$u_1 = r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2})$.







1. Start with coarse grid: $(\Delta x)_i = h_0 \ \forall \Delta_i$
2. Solve $E_h(u_h, v_h) \to \min_{\mathcal{C}_h}$
3. Compute error indicator $\eta(\Delta_i)$
4. $\eta(\Delta_i) > \delta \eta_{max} \implies$ refine
5. GO TO 2.

Adaptive grid refinement (monotone problems)

Gradient recovery in Ω as in Carstensen, Liu, Yan (Math. Comp. 2006)
Residual type estimator on $\partial\Omega$

z node, φ_z nodal basis function, $K_{j,z}$ triangle containing z, $\sum_{j=1}^{N_z} \alpha_{j,z} = 1$

$$G_h u_h = \sum_z (G_h v_h)(z)\, \varphi_z, \ (G_h v_h)(z) = \sum_{j=1}^{N_z} \alpha_{j,z} \nabla u_h|_{K_{j,z}}$$

A posteriori estimate (monotone, $p \geq 2$)

Theorem (with $G_{p,\delta}(x,y) = |y|^2[(|x|+|y|)^\delta(1+|x|+|y|)^{1-\delta}]^{p-2}$)

$$\|\hat{u}-\hat{u}_h, \hat{v}-\hat{v}_h\|_{X^p}^p \lesssim \eta_\Omega^2 + \eta_f^2 + \eta_S^2 + \eta_\partial^2 + \eta_g^2 \ , \quad \textit{where}$$

$$\eta_\Omega^2 = \sum_{K\in\mathcal{T}_h} \int_K G_{p,\delta}(\nabla\hat{u}_h, \nabla\hat{u}_h - G_h\hat{u}_h),$$

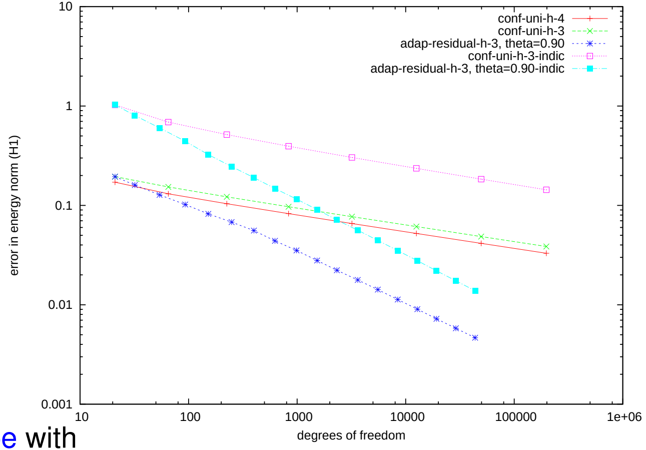
$$\eta_f^2 = \sum_{K\in\mathcal{T}_h} \int_K G_{p',1}(|\nabla\hat{u}_h|^{p-1}, h_K(f-f_K))$$

$$\eta_S^2 = \sum_{E\subset\partial\Omega} h_E \,\|\partial_s\{\mathcal{V}\hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0)\}\|_{L^2(E)}^2$$

$$\eta_\partial^2 = \sum_{E\subset\partial\Omega} h_E \,\|t_0 - \nu\cdot W'(\nabla\hat{u}_h) - \mathcal{W}(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) - (\mathcal{K}'-1)\hat{\phi}_h\|_{L^2(E)}^2$$

$$\eta_g^2 = \|(|\nu\cdot W'(\nabla\hat{u}_h)| - g)_+\|_{\widetilde{W}^{-\frac{1}{2},2}(\Gamma_s)}^{p'}$$

$$+ \int_{\Gamma_s} (|\nu\cdot W'(\nabla\hat{u}_h)| - g)_- |\hat{v}_h| + 2\int_{\Gamma_s} (\nu\cdot W'(\nabla\hat{u}_h)\hat{v}_h)_- \ .$$



L-shape with
$\Omega = [-\frac{1}{4}, \frac{1}{4}]^2 \backslash [0, \frac{1}{4}]^2$, $\Gamma_s = \overline{(-\frac{1}{4}, -\frac{1}{4})(\frac{1}{4}, -\frac{1}{4})} \cup \overline{(-\frac{1}{4}, -\frac{1}{4})(-\frac{1}{4}, \frac{1}{4})}$
$W'(t) = (\varepsilon + |t|)^{p-2} t$, with $p = 3$ and $\varepsilon = 10^{-5}$, $\Gamma_s = \emptyset$,
$u_1 = r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2})$, $u_2 = 0$. Uzawa with $\rho = 25$, Newton's method in every Uzawa-iteration step

Uzawa algorithm

1. Choose $\sigma_h^0 \in \Lambda_h = \{\sigma_h \in \widetilde{H}_h^{\frac{1}{2}}(\Gamma_s) \,:\, |\sigma_h(x)| \leq 1$ a.e. on $\Gamma_s\}$,.
2. For $n = 0, 1, 2, \ldots$ find $(u_h^n, v_h^n) \in X_h^p$ such that
   $\langle G' u_h^n, u_h \rangle + \langle S_h(u_h^n|_{\partial\Omega} + v_h^n), u_h|_{\partial\Omega} + v_h \rangle + \int_{\Gamma_s} g \sigma_h^n v_h \, ds = \lambda_h(u_h, v_h)$
   for all $(u_h, v_h) \in X_h^p$.
3. $\sigma_h^{n+1} = P_\Lambda(\sigma_h^n + \rho g v_h^n), \ \rho > 0, \quad P_\Lambda(\delta) = \sup\{-1, \inf(1, \delta)\}$
4. Repeat with 2. until a convergence criterion is satisfied.

DOF	$E_h(\hat{u}_h, \hat{v}_h)$	δE_h	α_{E_h}	It_{Uzawa}	$\tau(s)$
28	-0.511609	0.017249	—	2	0.190
80	-0.517938	0.010920	-0.435	2	0.640
256	-0.521857	0.007001	-0.382	2	2.440
896	-0.524293	0.004566	-0.341	2	11.05
3328	-0.525841	0.003017	-0.316	2	61.85
12800	-0.526865	0.001993	-0.308	2	437.5
50176	-0.527571	0.001287	-0.320	2	4218.

Table: Convergence rates and Uzawa steps for uniform meshes

Nonconvex functionals

Example

$$E_{simple}(u) = \int_0^1 dx \ (|\partial_x u(x)|^2 - 1)^2$$

$\inf\{E_{simple}(u) \ : \ u \in H_0^1(0,1)\} = 0,$ ∞-many minimizers

$\inf\{E_{simple}(u) + \int_0^1 dx \ |u(x)|^2 \ : \ u \in H_0^1(0,1)\} = 0,$ no minimizers

But: u_n minimizing sequence $\Rightarrow$
$\|u_n\|_{L^2} \to 0$, $\partial_x u_n \sim \pm 1$ (with probability $\frac{1}{2}$ each)

Convexification stable, suffices for some macroscopic properties

$$E^{**}_{simple}(u) = \int_0^1 dx \ (|\partial_x u(x)|^2 - 1)^2 \mathbf{1}_{\{|\partial_x u(x)| \geq 1\}}$$

Double-well potential: many minimizers

Convexification Φ^{**}: $\exists$ minimizer, possibly not unique

Lemma

a) $\{$*minimizers* $(\hat{u}, \hat{v})\} \neq \emptyset$, *bounded in* X^4

b) independent of minimizer $(\hat{u}, \hat{v})$*:*

1. *boundary value* $\hat{u}|_{\partial\Omega} + \hat{v}$
2. *region of microstructure* $\{x \in \Omega : |\nabla \hat{u}(x) + (\frac{1}{2}, -\frac{1}{2})|^2 > \frac{1}{2}\}$
3. *stress* $DW^{**}(\hat{u})$
4. *projected gradient* $\mathbb{P}\nabla \hat{u}$

These quantities can be computed numerically!

A priori estimate (double–well)

Theorem

$$
\begin{aligned}
&\|\sigma-\sigma_h\|^2_{L^{\frac{4}{3}}(\Omega)} + \|(\hat{u}-\hat{u}_h)|_{\partial\Omega} + \hat{v}-\hat{v}_h\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\quad + \|\mathbb{P}\nabla\hat{u} - \mathbb{P}\nabla\hat{u}_h\|^2_{L^2(\Omega)} + \|(\xi+\xi_h)^{1/2} A\nabla(\hat{u}-\hat{u}_h)\|^2_{L^2(\Omega)} + \|(\xi-\xi_h)\|^2_{L^2(\Omega)} \\
&\quad \leq \ C \inf_{(U_h,V_h)\in\mathcal{A}_h} \Big\{ \|u-U_h\|_{W^{1,4}(\Omega)} + \|(u-U_h)|_{\partial\Omega} + v - V_h)\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\qquad\qquad + \ \textit{(discretization error of S)} \ .
\end{aligned}
$$

Simple a posteriori estimate (double–well)

Theorem

$$\|\sigma - \sigma_h\|^2_{L^{\frac{4}{3}}(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\mathbb{P}\nabla\hat{u} - \mathbb{P}\nabla\hat{u}_h\|^2_{L^2(\Omega)}$$
$$+\|(\xi + \xi_h)^{1/2} A\nabla(\hat{u} - \hat{u}_h)\|^2_{L^2(\Omega)} + \|\xi - \xi_h\|^2_{L^2(\Omega)}$$
$$\lesssim \eta_\Omega + \eta_C + \eta_S^2 + \eta_\partial^2 \,,$$

where

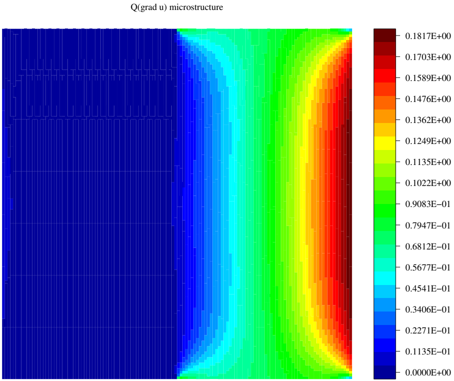
$$\eta_\Omega = \sum_K h_K \|f\|_{L^{4/3}(K)} + \sum_{E\cap\partial\Omega=\emptyset} h_E \|[\nu_E \cdot \sigma_h]\|_{L^2(E)} \,,$$

$$\eta_C = \eta_{C,1} + \eta_{C,2} = \sum_{E\subset\Gamma_s} \|(\nu_E \cdot \sigma_h)_+\|_{L^2(E)} + \sum_{E\subset\Gamma_s} \int_E (\nu_E \cdot \sigma_h)_- \, \hat{v}_h \,,$$

$$\eta_S^2 = \sum_{E\subset\partial\Omega} h_E \, \|\partial_s\{\mathcal{V}\hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0)\}\|^2_{L^2(E)}$$

$$\eta_\partial^2 = \sum_{E\subset\partial\Omega} h_E \, \|t_0 - \nu \cdot W'(\nabla\hat{u}_h) - \mathcal{W}(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) - (\mathcal{K}' - 1)\hat{\phi}_h\|^2_{L^2(E)}$$

Indicator of microstructure (non-adaptive)



indicator of microstructure:

$$Q(\nabla \hat{u}_h) = \max\{0, |\nabla \hat{u}_h + (\tfrac{1}{2}, -\tfrac{1}{2})|^2 - \tfrac{1}{2}\}$$

$$\Omega = (0,1)^2,\ \Gamma_s = \emptyset$$

(minimization of $\Phi^{**}(u) + \int_\Omega |\widetilde{f} - u|^2$

as in Carstensen/Plecháč 1998)

Recall: Linear elasticity

$\Omega^c \subset \mathbb{R}^d$, $d = 2, 3$

displacement from equilibrium $u : \Omega^c \to \mathbb{R}^d$

Hooke's law: energy is quadratic in u

$$\text{Energy}(u) = \frac{1}{2} \int_{\Omega^c} \varepsilon(u) : \mathcal{C}\varepsilon(u) \, dx$$

- Here $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. For d=2: $\nabla u = \begin{pmatrix} \partial_{x_1} u_1 & \partial_{x_2} u_1 \\ \partial_{x_1} u_2 & \partial_{x_2} u_2 \end{pmatrix}$.
- Stress $\sigma(u) := \mathcal{C}\varepsilon(u) = \sum_{k,l=1}^3 \mathcal{C}_{ijkl} \, \varepsilon_{kl}$
- The integrand is $\sum_{i,j,k,l=1}^3 \varepsilon_{ij} \, \mathcal{C}_{ijkl} \, \varepsilon_{kl}$.

High–order methods: Mixed hp–BEM in elasticity

Focus on integral equation: unilateral contact (i.e. $u \equiv 0$ in $\Omega \subset \mathbb{R}^2$)

- displacement u, stress $\sigma = \sigma(u)$
- non–penetration (Signorini) condition on normal components

$$\sigma_n := n \cdot \sigma \cdot n \leq 0, \; u_n := u \cdot n \leq g, \; \sigma_n(u_n - g) = 0$$

- friction (Tresca) condition on tangential components $\sigma_t := t \cdot \sigma \cdot n$

$$|\sigma_t| \leq \mathcal{F}, \; \sigma_t u_t + \mathcal{F}\,|u_t| = 0$$

- Dirichlet Γ_D, Neumann Γ_N, $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$, $\overline{\Gamma}_\Sigma := \overline{\Gamma}_N \cup \overline{\Gamma}_C$

Contact problem as saddle point problem ($\lambda = -\sigma(u)n$):
Find $(u, \lambda) \in \tilde{H}^{1/2}(\Gamma_\Sigma)^2 \times M^+(\mathcal{F})$ s.t.

$$\begin{aligned} \langle Su, v\rangle_{\Gamma_\Sigma} + \langle \lambda, v\rangle_{\Gamma_C} &= \langle f, v\rangle_{\Gamma_N} && \forall v \in \tilde{H}^{1/2}(\Gamma_\Sigma)^2 \\ \langle u, \mu - \lambda\rangle_{\Gamma_C} &\leq \langle g, \mu_n - \lambda_n\rangle_{\Gamma_C} && \forall \mu \in M^+(\mathcal{F}). \end{aligned}$$

$$M^+(\mathcal{F}) = \left\{ \mu \in \tilde{H}^{-1/2}(\Gamma_C)^2 : \langle \mu, v\rangle_{\Gamma_C} \leq \langle \mathcal{F}, |v_t|\rangle_{\Gamma_C} \, \forall v \in \tilde{H}^{1/2}(\Gamma_\Sigma)^2, v_n \leq 0 \right\}$$

Mixed formulation = saddle–point problem

Find $(u, \lambda) \in \tilde{H}^{1/2}(\Gamma_\Sigma)^2 \times M^+(\mathcal{F})$ s.t.

$$\begin{aligned} \langle Su, v\rangle_{\Gamma_\Sigma} + \langle \lambda, v\rangle_{\Gamma_C} &= \langle f, v\rangle_{\Gamma_N} && \forall v \in \tilde{H}^{1/2}(\Gamma_\Sigma)^2 \\ \langle u, \mu - \lambda\rangle_{\Gamma_C} &\leq \langle g, \mu_n - \lambda_n\rangle_{\Gamma_C} && \forall \mu \in M^+(\mathcal{F}). \end{aligned}$$

$$M^+(\mathcal{F}) = \left\{ \mu \in \tilde{H}^{-1/2}(\Gamma_C)^2 : \langle \mu, v\rangle_{\Gamma_C} \leq \langle \mathcal{F}, |v_t|\rangle_{\Gamma_C} \, \forall v \in \tilde{H}^{1/2}(\Gamma_\Sigma)^2, v_n \leq 0 \right\}$$

Uniquely solvable if the following operator is nondegenerate:

$$\begin{pmatrix} \langle S\cdot, \cdot\rangle_{\Gamma_\Sigma} & \langle \cdot, \cdot\rangle_{\Gamma_C} \\ \langle \cdot, \cdot\rangle_{\Gamma_C} & 0 \end{pmatrix}$$

inf–sup condition necessary and sufficient:

$$\tilde{\beta} \, \|\mu\|_{\tilde{H}^{-1/2}(\Gamma_C)} \leq \sup_{v \in \tilde{H}^{1/2}(\Gamma_\Sigma)^2 \setminus \{0\}} \frac{\langle \mu, v\rangle_{\Gamma_C}}{\|v\|_{\tilde{H}^{1/2}(\Gamma_\Sigma)}} \qquad \forall \mu \in \tilde{H}^{-1/2}(\Gamma_C)^2 \, .$$

Discretization

Find $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M_{kq}^{+}(\mathcal{F})$ s.t.

$$\begin{aligned} \left\langle S_{hp} u^{hp}, v^{hp} \right\rangle_{\Gamma_\Sigma} + \left\langle \lambda^{kq}, v^{hp} \right\rangle_{\Gamma_C} &= \left\langle f, v^{hp} \right\rangle_{\Gamma_N} & \forall v^{hp} \in \mathcal{V}_{hp} \\ \left\langle \mu^{kq} - \lambda^{kq}, u^{hp} \right\rangle_{\Gamma_C} &\leq \left\langle g, \mu_n^{kq} - \lambda_n^{kq} \right\rangle_{\Gamma_C} & \forall \mu^{kq} \in M_{kq}^{+}(\mathcal{F}) \ . \end{aligned}$$

Two meshes on the boundary $\mathcal{T}_h$, $\hat{\mathcal{T}}_k$

$$\mathcal{V}_{hp} = \{ v^{hp} \in C^0(\Gamma_\Sigma)^2 \cap \tilde{H}^{1/2}(\Gamma_\Sigma)^2 : v^{hp}|_E \in \left[\mathbb{P}_{p_E}\right]^2, \ v_{hp} = 0 \text{ at } \partial\Gamma_\Sigma \}$$

$$M_{kq}^{+}(\mathcal{F}) = \{ \mu^{kq} : \mu^{kq}|_E \in \left[\mathbb{P}_{q_E}\right]^2, \ \mu_n^{kq} \geq 0, \ |\mu_t^{kq}(x)| \leq \mathcal{F}(x) \text{ for } x \in G_{kq} \}$$

G_{kq} Gauss points, $M_{kq}^{+}(\mathcal{F}) \not\subset M^{+}(\mathcal{F})$.

Discretization

Find $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$ s.t.

$$\left\langle S_{hp} u^{hp}, v^{hp} \right\rangle_{\Gamma_\Sigma} + \left\langle \lambda^{kq}, v^{hp} \right\rangle_{\Gamma_C} = \left\langle f, v^{hp} \right\rangle_{\Gamma_N} \qquad \forall v^{hp} \in \mathcal{V}_{hp}$$

$$\left\langle \mu^{kq} - \lambda^{kq}, u^{hp} \right\rangle_{\Gamma_C} \le \left\langle g, \mu^{kq}_n - \lambda^{kq}_n \right\rangle_{\Gamma_C} \qquad \forall \mu^{kq} \in M^+_{kq}(\mathcal{F}) \ .$$

$$\mathcal{V}_{hp} = \{ v^{hp} \in C^0(\Gamma_\Sigma)^2 \cap \tilde{H}^{1/2}(\Gamma_\Sigma)^2 : v^{hp}|_E \in \left[\mathbb{P}_{p_E}\right]^2, \ v_{hp} = 0 \text{ at } \partial\Gamma_\Sigma \}$$

$$M^+_{kq}(\mathcal{F}) = \{ \mu^{kq} : \mu^{kq}|_E \in \left[\mathbb{P}_{q_E}\right]^2, \ \mu^{kq}_n \ge 0, \ |\mu^{kq}_t(x)| \le \mathcal{F}(x) \text{ for } x \in G_{kq} \}$$

Discretization uniquely solvable if the following operator is nondegenerate:

$$\begin{pmatrix} \langle S_{hp} \cdot, \cdot \rangle_{\Gamma_\Sigma} & \langle \cdot, \cdot \rangle_{\Gamma_C} \\ \langle \cdot, \cdot \rangle_{\Gamma_C} & 0 \end{pmatrix}$$

True when $\mathcal{T}_h$ much finer than $\hat{\mathcal{T}}_k$ (Babuska, Schröder, ...)

Discretization

Find $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$ s.t.

$$\begin{aligned} \left\langle S_{hp} u^{hp}, v^{hp} \right\rangle_{\Gamma_\Sigma} + \left\langle \lambda^{kq}, v^{hp} \right\rangle_{\Gamma_C} &= \left\langle f, v^{hp} \right\rangle_{\Gamma_N} && \forall v^{hp} \in \mathcal{V}_{hp} \\ \left\langle \mu^{kq} - \lambda^{kq}, u^{hp} \right\rangle_{\Gamma_C} &\leq \left\langle g, \mu^{kq}_n - \lambda^{kq}_n \right\rangle_{\Gamma_C} && \forall \mu^{kq} \in M^+_{kq}(\mathcal{F}) \ . \end{aligned}$$

$$\mathcal{V}_{hp} = \{ v^{hp} \in C^0(\Gamma_\Sigma)^2 \cap \tilde{H}^{1/2}(\Gamma_\Sigma)^2 : v^{hp}|_E \in [\mathbb{P}_{p_E}]^2 \,,\ v_{hp} = 0 \text{ at } \partial\Gamma_\Sigma \}$$

$$M^+_{kq}(\mathcal{F}) = \{ \mu^{kq} : \mu^{kq}|_E \in [\mathbb{P}_{q_E}]^2 \,,\ \mu^{kq}_n \geq 0,\ |\mu^{kq}_t(x)| \leq \mathcal{F}(x) \text{ for } x \in G_{kq} \}$$

Idea: The following operator would be nondegenerate for large δ

$$\begin{pmatrix} \left\langle S_{hp} \cdot, \cdot \right\rangle_{\Gamma_\Sigma} & \left\langle \cdot, \cdot \right\rangle_{\Gamma_C} \\ \left\langle \cdot, \cdot \right\rangle_{\Gamma_C} & -\delta \end{pmatrix}$$

But: Solution very different.

Discretization

Find $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$ s.t.

$$\left\langle S_{hp} u^{hp}, v^{hp} \right\rangle_{\Gamma_\Sigma} + \left\langle \lambda^{kq}, v^{hp} \right\rangle_{\Gamma_C} - \left\langle \gamma \left(\lambda^{kq} + S_{hp} u^{hp} \right), S_{hp} v^{hp} \right\rangle_{\Gamma_C} = \left\langle f, v^{hp} \right\rangle_{\Gamma_N}$$
$$\left\langle \mu^{kq} - \lambda^{kq}, u^{hp} \right\rangle_{\Gamma_C} - \left\langle \gamma \left(\mu^{kq} - \lambda^{kq} \right), \lambda^{kq} + S_{hp} u^{hp} \right\rangle_{\Gamma_C} \leq \left\langle g, \mu^{kq}_n - \lambda^{kq}_n \right\rangle$$

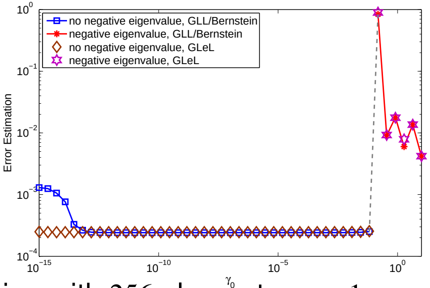
with $\gamma = \gamma_0 \frac{h}{p^2}$. $\lambda = -Su \rightsquigarrow$ new terms ~ 0.

$$M^+_{kq}(\mathcal{F}) = \{ \mu^{kq} : \mu^{kq}|_E \in [\mathbb{P}_{q_E}]^2, \ \mu^{kq}_n \geq 0, \ |\mu^{kq}_t(x)| \leq \mathcal{F}(x) \text{ for } x \in G_{kq} \}$$

Discretization uniquely solvable for small $\gamma_0 > 0$:

$$\begin{pmatrix} \langle S_{hp} \cdot, \cdot \rangle_{\Gamma_\Sigma} - \langle \gamma S_{hp} \cdot, S_{hp} \cdot \rangle_{\Gamma_C} & \langle \cdot, \cdot \rangle_{\Gamma_C} - \langle \gamma S_{hp} \cdot, \cdot \rangle_{\Gamma_C} \\ \langle \cdot, \cdot \rangle_{\Gamma_C} - \langle \gamma S_{hp} \cdot, \cdot \rangle_{\Gamma_C} & - \langle \gamma S_{hp} \cdot, \cdot \rangle_{\Gamma_C} \end{pmatrix}$$

Error estimator vs. stabilization parameter γ_0



uniform discretisation with 256 elements, $p = 1$

$\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$, $\Gamma_C = [-\frac{1}{2}, \frac{1}{2}] \times \{-\frac{1}{2}\}$, $\Gamma_N = \partial\Omega \setminus \Gamma_C$

Elasticity parameters $E = 5$, $\nu = 0.45$, friction coefficient 0.3.

$$t_{\text{side}} = \begin{pmatrix} -10\,\mathrm{sign}(x_1)(\frac{1}{2}+x_2)(\frac{1}{2}-x_2)\exp(-10(x_2+\frac{4}{10})^2) \\ \frac{7}{8}(\frac{1}{2}+x_2)(\frac{1}{2}-x_2) \end{pmatrix}$$

$$t_{\text{top}} = \begin{pmatrix} 0 \\ -\frac{25}{2}(\frac{1}{2}-x_1)^2(\frac{1}{2}+x_1)^2 \end{pmatrix}$$

Convergence

Main challenges:

- stabilization is not exact: $\lambda = -Su \neq -S_{hp}u$
- nonconforming: $M^+_{kq}(\mathcal{F}) \not\subset M^+(\mathcal{F})$

Theorem (a priori error estimate)

Assume $(u,\lambda) \in H^{1+\alpha}(\Gamma)^2 \times H^{\alpha}(\Gamma_C)^2 \cap C^0(\Gamma_C)^2$, $\alpha \in [0, \frac{1}{2})$, $h = k$, $p = q + 1$.

$$\rightsquigarrow \|u - u^{hp}\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} + \|\gamma^{\frac{1}{2}}(\lambda - \lambda^{hq})\|^2_{L^2(\Gamma_C)} + \|\psi - \psi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} \lesssim h^{\alpha/2} p^{-\alpha/3}$$

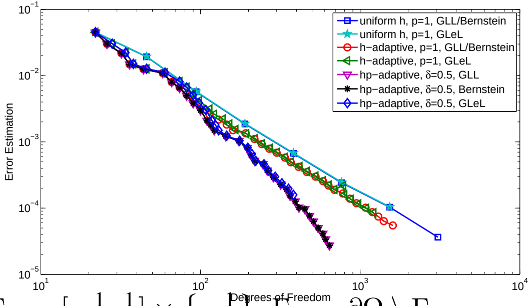
- intricate proof, more complicated rate for arbitrary discretisations
- compare Hild, Lleras, Renard for FEM, rates not optimal
- rate dominated by $S \neq S_{hp}$

Computable error estimates

Theorem (a posteriori error estimate)

$$
\begin{aligned}
&\|u - u^{hp}\|^2_{\tilde{H}^{1/2}(\Gamma_\Sigma)} + \|\psi - \psi^{hp}\|^2_{H^{-1/2}(\Gamma)} + \|\lambda^{kq} - \lambda\|^2_{\tilde{H}^{-1/2}(\Gamma_C)} \\
&\leq \sum_{E \in \mathcal{T}_h|_{\Gamma_N}} \frac{h_E}{p_E} \left\|f - S_{hp} u^{hp}\right\|^2_{L^2(E)} + \sum_{E \in \mathcal{T}_h|_{\Gamma_C}} \frac{h_E}{p_E} \left\|\lambda^{kq} + S_{hp} u^{hp}\right\|^2_{L^2(E)} \\
&+ \sum_{E \in \mathcal{T}_{h,\Gamma}} h_E \left\| \frac{\partial}{\partial s} \left(V\psi^{hp} - (K + \frac{1}{2}) u^{hp} \right) \right\|^2_{L^2(E)} + \left\langle \left(\lambda_n^{kq}\right)^+, \left(g - u_n^{hp}\right)^+ \right\rangle_{\Gamma_C} \\
&+ \left\| \left(g - u_n^{hp}\right)^- \right\|^2_{H^{1/2}(\Gamma_C)} + \left\| \left(\lambda_n^{kq}\right)^- \right\|^2_{\tilde{H}^{-1/2}(\Gamma_C)} \\
&+ \left\| \left(\left|\lambda_t^{kq}\right| - \mathcal{F} \right)^+ \right\|^2_{\tilde{H}^{-1/2}(\Gamma_C)} - \left\langle \left(\left|\lambda_t^{kq}\right| - \mathcal{F} \right)^-, \left|u_t^{hp}\right| \right\rangle_{\Gamma_C} \\
&- \left\langle \lambda_t^{kq}, u_t^{hp} \right\rangle_{\Gamma_C} + \left\langle \left|\lambda_t^{kq}\right|, \left|u_t^{hp}\right| \right\rangle_{\Gamma_C}
\end{aligned}
$$

Convergence of stabilized mixed h and hp–BEM



$\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$, $\Gamma_C = [-\frac{1}{2}, \frac{1}{2}] \times \{-\frac{1}{2}\}$, $\Gamma_N = \partial\Omega \setminus \Gamma_C$

Elasticity parameters $E = 5$, $\nu = 0.45$, friction coefficient 0.3.

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$$t_{\text{top}} = \begin{pmatrix} 0 \\ -\frac{25}{2}(\frac{1}{2}-x_1)^2(\frac{1}{2}+x_1)^2 \end{pmatrix}$$

Summary & References

Efficient FE–BE analysis in (non–Hilbert) $L^p - L^r$ spaces

Adaptive methods for strongly nonlinear coupling problems

Stabilized mixed high–order methods for integral equations

Two references:

HG, M. Maischak, E. Schrohe, E. P. Stephan, *Adaptive FE–BE coupling for strongly nonlinear transmission problems with Coulomb friction*, Numer. Math. (2011), 26 pages.

L. Banz, HG, A. Issaoui, E. P. Stephan, *Stabilized mixed hp-BEM for frictional contact problems in linear elasticity*, Numer. Math. (2016), 47 pages.