Interface and contact problems

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Which equations describe materials in contact?

Basic principle: Energy is additive

For a system composed of two subsystems

$$\text{Total Energy} = \text{Energy of 1} + \text{Energy of 2} + \text{Interaction betw. 1/2.}$$

In elastic problems, the interaction might correspond to friction between $1/2$, "surface energy" etc. $\rightsquigarrow$ *localized on* Γ_c =contact area

$$(\text{Interaction betw. 1/2}) = \int_{\Gamma_C} \cdots \, dx$$

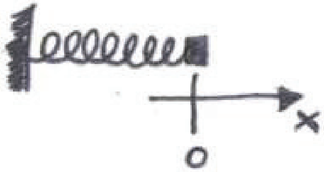
Nature minimizes Total Energy over physically allowed configurations. In this course this will lead to

unconstrained, smooth variational problems $\rightsquigarrow$ PDE

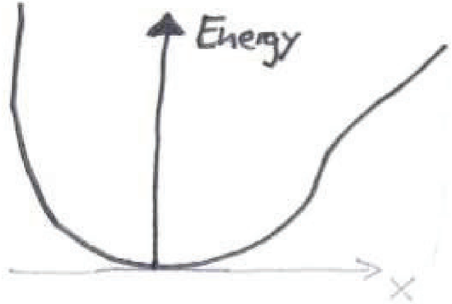
variational problems not C^1 or constrained $\rightsquigarrow$ variational inequalities

Energy

Hooke's law:



$\text{Energy}(x) \simeq \frac{1}{2} k x^2$



- Approximation for small $|x|$
- not useful for large $|x|$ or if $k \to 0$.

Continuous media

Energy

$$\simeq \frac{1}{2}\delta\vec{x} \cdot \underbrace{C(\nabla u + \nabla u^T)}_{\text{matrix}} \delta\vec{x}$$

material at rest (reference)

initial positions $\vec{x}$, $\vec{x}+\delta\vec{x}$

deformed material

deformed positions $\vec{x}+\vec{u}(\vec{x})$
$\vec{x}+\delta\vec{x}+\vec{u}(\vec{x}+\delta\vec{x})$

$\vec{u}(\vec{x})$ deformation

Total energy

$$\text{Energy}(u) \simeq \frac{1}{2}\int_\Omega \varepsilon(u) : \mathcal{C}\varepsilon(u)\, dx = \int_\Omega \mathcal{W}(\varepsilon(u(x)))\, dx$$

- Here $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, $\nabla u = \begin{pmatrix} \partial_{x_1} u_1 & \partial_{x_2} u_1 & \partial_{x_3} u_1 \\ \partial_{x_1} u_2 & \partial_{x_2} u_2 & \partial_{x_3} u_2 \\ \partial_{x_1} u_3 & \partial_{x_2} u_3 & \partial_{x_3} u_3 \end{pmatrix}$.
- The integrand is $\sum_{i,j,k,l=1}^{3} \varepsilon_{ij}\, \mathcal{C}_{ijkl}\, \varepsilon_{kl}$.
- Large $\nabla u \rightsquigarrow$ quadratic approximation bad.

Properties of energy density $\leftrightsquigarrow$ physics

$\mathcal{W}(\varepsilon(u))$ convex function $\leftrightsquigarrow$ larger deformations require larger force
$\mathcal{W}(\varepsilon(u))$ nonconvex function: phase transitions!

Nature minimizes energy

For simplicity, we will mostly consider a toy problem:

- $u : \Omega \to \mathbb{R}$ scalar function.
- ∇u instead of $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$.
- $\rightsquigarrow$ simplest energy (with force f) is given by

$$\text{Energy}(u) = \frac{1}{2}\int_\Omega (D(x)\nabla u(x)) \cdot \nabla u(x)\, dx - \int_\Omega f(x)u(x)\, dx$$

Nature finds (local) minimizers of the energy:

$$\frac{d\text{Energy}}{du} = 0$$

What does $\frac{d\text{Energy}}{du} = 0$ mean?
For all “physically” reasonable $h : \Omega \to \mathbb{R}$

$$\frac{d}{dt}\Big|_{t=0} E(u + th) = 0$$

Nature minimizes energy $\rightsquigarrow$ differential equations

$$\text{Energy}(u) = \frac{1}{2}\int_\Omega (D\nabla u)\cdot \nabla u \, dx - \int_\Omega fu \, dx$$

$\Omega \subset \mathbb{R}^n$ bounded, force f, both nice. $D = 1$, n unit normal vector to $\partial\Omega$.

Theorem

u minimizes $\text{Energy}(v) = \frac{1}{2}\int_\Omega (\nabla v)^2 - \int_\Omega fv$ over $H^1(\Omega)$

$\iff u \in H^1(\Omega)$ solves $-\Delta v = f, \quad \frac{\partial v}{\partial n}\big|_{\partial\Omega} = 0$

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Proof: $\forall h \in H^1(\Omega)$

$$\begin{aligned} E(u+h) - E(u) &= \frac{1}{2}\int_\Omega (\nabla u + \nabla h)^2 - \int_\Omega f(u+h) - \frac{1}{2}\int_\Omega (\nabla u)^2 + \int_\Omega fu \\ &= \int_\Omega \nabla u \nabla h - \int_\Omega fh + \frac{1}{2}\int_\Omega (\nabla h)^2 \\ &= -\int_\Omega (\Delta u + f)h + \int_{\partial\Omega} (n\cdot\nabla u)h + \frac{1}{2}\int_\Omega (\nabla h)^2 \end{aligned}$$

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"$\Longrightarrow$" If $-\Delta u = f$, $\frac{\partial v}{\partial n}\big|_{\partial\Omega} = 0 \Rightarrow E(u+h) - E(u) = 0 + \frac{1}{2}\int_\Omega (\nabla h)^2 \geq 0$

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"$\Longleftarrow$"

$E(u+\lambda h) - E(u) = -\lambda\left(\int_\Omega (\Delta u + f)h + \int_{\partial\Omega}(n\cdot\nabla u)h\right) + \frac{\lambda^2}{2}\int_\Omega (\nabla h)^2 \geq 0.$

For all $\lambda \in \mathbb{R} \Rightarrow \int_\Omega (\Delta u + f)h + \int_{\partial\Omega}(n\cdot\nabla u)h = 0.$

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For all $\lambda \in \mathbb{R} \Rightarrow \int_\Omega (\Delta u + f)h + \int_{\partial\Omega}(n\cdot\nabla u)h = 0.$

For all $h \in H^1_0 \Rightarrow \int_\Omega (\Delta u + f)h = 0$, therefore $\Delta u + f = 0$ a.e.

Thus, for all $h \in H^1 \Rightarrow \int_{\partial\Omega}(n\cdot\nabla u)h = 0$, therefore $n\cdot\nabla u = 0$ a.e.

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inhomogeneous/anisotropic energy:

$$E(u) = \frac{1}{2}\int_\Omega (\nabla u(x))^t D(x) \nabla u(x) - \int_\Omega fu,\ D = D^t > 0 \rightsquigarrow -\mathrm{div}\,(D\nabla u) = f$$

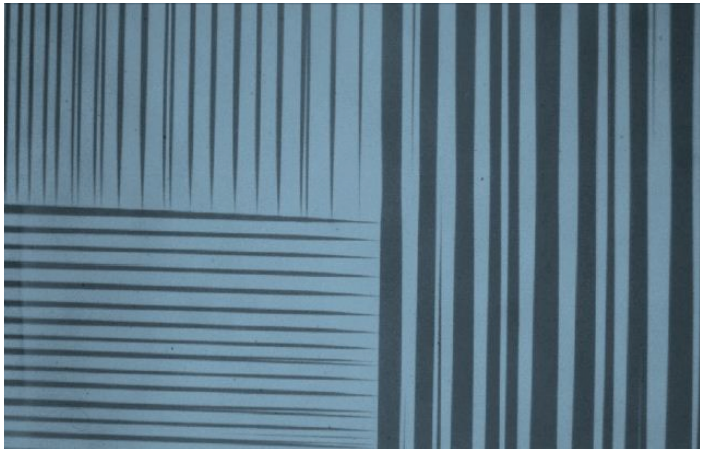
Examples of nonlinear operators

$$E(u) = \frac{1}{2}\int_\Omega (\nabla u(x))^t D(x)\nabla u(x) - \int_\Omega fu \rightsquigarrow -\mathrm{div}\,(D\nabla u) = f$$

p-Laplace: $E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega fu \rightsquigarrow -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = f$, $p \in (1,\infty)$

- $p < 2$ hair gel, glaciers, $p > 2$ thick emulsion of sand and water

phase transitions / bistable materials: double–well potential

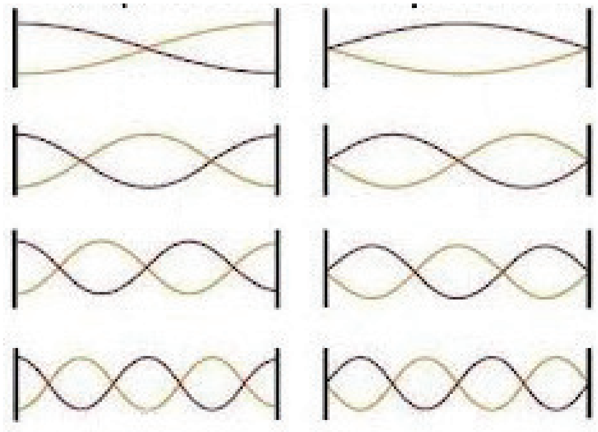


(from S. Müller)

$$E(u) = \int_\Omega |\nabla u - (1,0)|^2 \, |\nabla u - (0,1)|^2 - \int_\Omega fu$$

Boundary conditions

Neumann and Dirichlet:



Contact: Signorini (= nonpenetration, $\perp$ wall) and friction ($\parallel$ wall)