Penalty: $\quad a(u^\varepsilon, v) - \langle p^\varepsilon, v\rangle_{\Gamma_C} = \langle f, v\rangle \quad$ (PM) $\quad \forall v \in H^1_{\Gamma_D}$

$$p^\varepsilon := -\frac{1}{\varepsilon}(u^\varepsilon - \varphi)^+ \le 0$$

Mixed: (**) $\quad a(u, v) - \langle \lambda, v\rangle_{\Gamma_C} = \langle f, v\rangle \quad \forall v \in H^1_{\Gamma_D}$

$$\langle \mu - \lambda, u - \varphi\rangle_{\Gamma_C} \ge 0 \quad \forall \mu \in \Lambda$$

Observations: • $\mu = 2\lambda$ and $\mu = 0$ in $(**_2) \Rightarrow \langle \lambda, u - \varphi\rangle = 0$

• since (**) $\Leftrightarrow$ original formulation: $u \le \varphi$ (i.e. $u \in K$)

Lemma 1 $\quad \langle \lambda - p^\varepsilon, u - \varphi\rangle_{\Gamma_C} \le 0$

Proof: $\langle \lambda - p^\varepsilon, u - \varphi\rangle_{\Gamma_C} = -\langle \underbrace{p^\varepsilon}_{\le 0}, \underbrace{u - \varphi}_{\le 0}\rangle_{\Gamma_C} \le 0 \quad \square$

Lemma 2 $\quad \langle p^\varepsilon - \lambda, (u^\varepsilon - \varphi)^-\rangle_{\Gamma_C} \le 0$

Proof: $\langle p^\varepsilon - \lambda, (u^\varepsilon - \varphi)^-\rangle_{\Gamma_C} = \underbrace{\langle -\frac{1}{\varepsilon}(u^\varepsilon - \varphi)^+, (u^\varepsilon - \varphi)^-\rangle}_{= 0} - \langle \underbrace{\lambda}_{\le 0}, \underbrace{(u^\varepsilon - \varphi)^-}_{\le 0}\rangle_{\Gamma_C} \le 0 \quad \square$

$(**_1) - (\text{PM}) \Rightarrow a(u - u^\varepsilon, u - u^\varepsilon) = \langle \lambda - p^\varepsilon, u - u^\varepsilon\rangle_{\Gamma_C}$

Lemma 1 $= \underbrace{\langle \lambda - p^\varepsilon, u - \varphi\rangle_{\Gamma_C}}_{\le 0} - \langle \lambda - p^\varepsilon, u^\varepsilon - \varphi\rangle_{\Gamma_C}$

$\le \langle p^\varepsilon - \lambda, \underbrace{(u^\varepsilon - \varphi)^+}_{= -\varepsilon p^\varepsilon}\rangle + \underbrace{\langle (p^\varepsilon - \lambda), (u^\varepsilon - \varphi)^-\rangle}_{\le 0 \text{ (Lemma 2)}}$

$= \langle \lambda - p^\varepsilon, \varepsilon\lambda\rangle + \underbrace{\varepsilon \langle p^\varepsilon - \lambda, p^\varepsilon - \lambda\rangle}_{\text{add } \ge 0}$