When is the stiffness matrix an M-matrix?

Def: $A \in \mathbb{R}^{n\times n}$ is said to be an M-matrix if $\exists\, B \in \mathbb{R}^{n\times n}$, $B_{ij} \geq 0$ $\exists\, s \geq$ largest eigenvalue of B s.t. $A = s\mathbb{1} - B$.

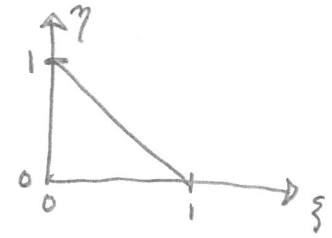
Let $\mathcal{T}_h$ be a triangulation of $\Omega \subseteq \mathbb{R}^2$, $H_h \subseteq H^1(\Omega)$ a subspace of piecewise linear functions on $\mathcal{T}_h$ with basis of hat functions $\{\phi_1, \dots, \phi_n\}$.

Let
$$A_{ij} = \int_\Omega \nabla_x \phi_i \cdot \nabla_x \phi_j \, dx .$$
be the associated stiffness matrix.

Lemma: If $\forall\, T \in \mathcal{T}_h$ the interior angles are acute, then $A = (A_{ij})$ is an M-matrix.

Proof: We assemble A from the local stiffness matrices A_T.

Recall the local basis functions and their gradients on the reference triangle



$$\bar{\Phi}_1(\xi,\eta) = 1 - \xi - \eta \quad , \quad \nabla_{(\xi,\eta)} \bar{\Phi}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$
$$\bar{\Phi}_2(\xi,\eta) = \xi \quad , \quad \nabla_{(\xi,\eta)} \bar{\Phi}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\bar{\Phi}_3(\xi,\eta) = \eta \quad , \quad \nabla_{(\xi,\eta)} \bar{\Phi}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since $\nabla_x = H^{-T} \nabla_{(\xi,\eta)}$ for some H, the local stiffness matrix is then

$$(A_T)_{ij} = \int_T \nabla_x \phi_i \cdot \nabla_x \phi_j = (\nabla_{(\xi,\eta)} \bar{\Phi}_i) H^{-1} H^{-T} (\nabla_{(\xi,\eta)} \bar{\Phi}_j)\, \mathrm{Area}(T)$$

The diagonal entries $(A_T)_{ii}$ are > 0.

We show that the off-diagonal entries A_{ij} are ≤ 0:

As A_T is symmetric, it suffices to prove this for $j>i$.

We first express the conditions on the angles of T in a more convenient form: Let p_0, p_1, p_2 be the vertices of T and

$\measuredangle(\overline{p_0p_1}, \overline{p_0p_2}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then

$$(0,1] \ni \cos(\measuredangle(\overline{p_0p_1}, \overline{p_0p_2})) = \frac{(p_0-p_1)\cdot(p_2-p_0)}{\|p_0-p_1\|\,\|p_2-p_0\|}, \text{ or}$$

$$(p_0-p_1)\cdot(p_2-p_0) = \begin{pmatrix} H_{21} \\ H_{11} \end{pmatrix}\cdot\begin{pmatrix} H_{22} \\ H_{12} \end{pmatrix} = H_{11}H_{12} + H_{21}H_{22} > 0 \quad (*)$$

Similarly, from $\measuredangle(\overline{p_1p_2}, \overline{p_1p_0}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we obtain

$$0 < (p_2-p_0+p_0-p_1)\cdot(p_0-p_1) = (p_2-p_0)(p_0-p_1) + \|p_0-p_1\|^2$$

$$= \begin{pmatrix} -H_{12} \\ -H_{22} \end{pmatrix}\cdot\begin{pmatrix} H_{11} \\ H_{21} \end{pmatrix} + \|p_0-p_1\|^2 = -H_{11}H_{12} - H_{21}H_{22} + H_{11}^2 + H_{21}^2 \quad (**)$$

Also recall the formula for the inverse of a 2×2 matrix:

$$H^{-T} = \frac{1}{\det H}\begin{pmatrix} H_{22} & -H_{21} \\ -H_{12} & H_{11} \end{pmatrix}$$

We now check that $(A_T)_{ij} < 0$ for $j>i$:

- $\begin{pmatrix} 1 \\ 0 \end{pmatrix} H^{-1} H^{-T} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(\frac{1}{\det H}\right)^2 \begin{pmatrix} H_{22} \\ -H_{12} \end{pmatrix}\cdot\left(-\begin{pmatrix} H_{21} \\ H_{11} \end{pmatrix}\right) = -\left(\frac{1}{\det H}\right)^2 (H_{11}H_{12} + H_{21}H_{22}) < 0$ by $(*)$

- $\begin{pmatrix} -1 \\ -1 \end{pmatrix} H^{-1} H^{-T} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(\frac{1}{\det H}\right)^2 \begin{pmatrix} -H_{22}+H_{21} \\ H_{12}-H_{11} \end{pmatrix}\cdot\begin{pmatrix} -H_{21} \\ H_{11} \end{pmatrix} = \left(\frac{1}{\det H}\right)^2 (-H_{11}^2 - H_{21}^2 + H_{11}H_{12} + H_{21}H_{22}) < 0$

- $\begin{pmatrix} -1 \\ -1 \end{pmatrix} H^{-1} H^{-T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{1}{\det H}\right)^2 \begin{pmatrix} -H_{22}+H_{21} \\ H_{12}-H_{11} \end{pmatrix}\cdot\begin{pmatrix} H_{22} \\ -H_{12} \end{pmatrix} = -\left(\frac{1}{\det H}\right)^2 \cdot (H_{12}^2 + H_{22}^2 - H_{11}H_{12} - H_{21}H_{22}) < 0$

$\Rightarrow (A_T)_{ij} < 0$ for $i \neq j$ $\Rightarrow A_{ij} \le 0$ for $i \neq j$. The assertion follows with $s = \max A_{ii}$.