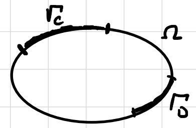
Lemma: Let the interior angles of all $T \in \mathcal{T}_h$ be $< \frac{\pi}{2}$
$\Rightarrow A$ is an M-matrix

2) Contact problems:

$$E(u) = \frac{1}{2}\, a(u,u) - \langle f,u\rangle + j(u) \longrightarrow \min_K$$

K closed convex $\subseteq H := \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D \}$



$$a(u,v) = \frac{1}{2}\int_\Omega \nabla u \cdot \nabla v$$

$$\langle f,u\rangle = \int_\Omega fu \quad , f \in L^2(\Omega)$$

$$j(u) = \int_{\Gamma_C} g|u| \quad , \quad 0 \le \underset{\text{friction}}{g} \in L^\infty(\Gamma_C)$$

Last week: Min. problem has unique solution, it is characterized by the variational inequality

(VI) Find $u \in K$: $a(u,v-u) + j(v) - j(u) \ge \langle f, v-u\rangle \quad \forall v \in K$

To solve this, we first eliminate the absolute value in j, using a Lagrange multiplier:

$$\Lambda := \{ \sigma \in L^2(\Gamma_C) : |\sigma| \le 1 \text{ a.e.} \}$$

$$q : \Lambda \times K \to \mathbb{R} \quad , q(\sigma,u) = \int_{\Gamma_C} g\sigma u \ .$$

i.e. $q(\operatorname{sign}(u), u) = j(u)$

An equivalent reformulation of (VI) is:

(VI_q) Find $(u,\sigma_u) \in K \times \Lambda$: $a(u,v-u) + q(\sigma_u, v-u) \ge \langle f, v-u\rangle \quad \forall v \in K$

and $\sigma_u u = |u|$ on Γ_C

(VI_q) comes from the following saddle point problem for $F(u,\sigma) := \frac{1}{2}a(u,u) - \langle f,u\rangle + q(\sigma,u)$:

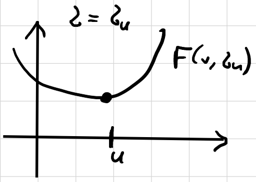
(S) Find $(u,\sigma_u) \in K \times \Lambda$ s.t. $\forall v \in K \ \forall \sigma \in \Lambda$

$$F(u,\sigma) \le F(u,\sigma_u) \le F(v,\sigma_u)$$

Lemma: (S) $\Leftrightarrow$ (VI_q)

Proof: „$\Rightarrow$": Let (u,σ_u) a solution to (S) $\Rightarrow \forall v \in K \ \forall \varepsilon \in (0,1)$

$$0 \le \frac{1}{\varepsilon}\left(F(u+\varepsilon(v-u), \sigma_u) - F(u,\sigma_u) \right)$$



$$= a(u, v-u) - \langle f, v-u\rangle + \underbrace{\int_{\Gamma_c} g\,\delta_u\,(v-u)}_{= q(\delta_u, v-u)} + \varepsilon(\ldots)$$

As $\varepsilon \to 0$: 1^{st} ineq. in (VI_q)

Further: $0 \le F(u, \delta_u) - F(u, \delta) = \int_{\Gamma_c} g(\delta_u - \delta)u \qquad \forall \delta \in \Lambda$

For $\delta = \mathrm{sign}(u) \in \Lambda$: $\quad 0 \le \int_{\Gamma_c} \underbrace{g}_{\ge 0} \underbrace{(\delta_u u - |u|)}_{\le 0 \text{ by def of } \Lambda}$

$\Rightarrow$ integrand $= 0$ a.e.: $\delta_u u = |u|$ on Γ_c, i.e. second ineq. in (VI_q)

"$\Leftarrow$": Let $(u, \delta_u) \in K \times \Lambda$ solution to (VI_q)

$$\Rightarrow F(v, \delta_u) \underbrace{- \tfrac{1}{2} a(v-u, v-u)}_{\le 0} \ge F(u, \delta_u) \qquad \forall v \in K$$

(Algebra $+ (VI_q)$) $\qquad \Rightarrow 2^{nd}$ ineq. in (S)

By definition of Λ and δ_u: $\forall \delta \in \Lambda$

$F(u,\delta) - F(u,\delta_u) = \int_{\Gamma_c} g(\delta - \delta_u)u = \int_{\Gamma_c} g(\delta u - |u|) \le 0 \qquad \Rightarrow 1^{st}$ ineq.

$\blacksquare$

We solve $(VI_q) \Leftrightarrow (S)$ for $(u, \delta_u) \in K \times \Lambda$ using, for example, the $\boxed{\text{Uzawa algorithm:}}$

- choose $\delta^0 \in \Lambda$, $\rho > 0$ sufficiently small (see later)

For $n = 1, 2, 3, \ldots$

- $u^n \in K$ s.t. $F(u^n, \delta^{n-1}) \le F(v, \delta^{n-1}) \quad \forall v \in K$

  $(\Leftrightarrow (*) \quad a(u^n, v-u^n) \ge \langle f, v-u^n\rangle - q(\delta^n, v-u^n) \quad \forall v \in K$

  If we have friction contact only (i.e. $K \subseteq H$ a subspace) substitute $v = u^n + w$, $w \in K = H$; $u^n - w$, $w \in K$

  $\Downarrow$

  $a(u^n, w) \ge \langle f, w\rangle - q(\delta^{n-1}, w)$

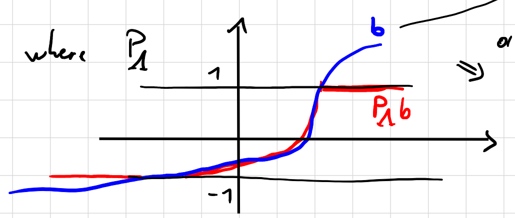
  $a(u^n, -w) \ge \langle f, -w\rangle - q(\delta^{n-1}, -w)$

  $\Rightarrow \quad - \; = \; -$

  Matrix equation!

- Set $\delta^n = P_\Lambda(\delta^{n-1} + \rho g u^n)$ where P_Λ

  $P_\Lambda b(x) = \max\{-1, \min\{1, b(x)\}\}$



- Stop if convergence criterion is satisfied

Theorem: Let $0 < \rho < 2\alpha \|g\|^{-2}_{L^\infty(\Gamma_c)}$. Then $\forall\, \lambda^0 \in \Lambda$ the Uzawa algorithm converges in Norm: $\|u^n - u\|_H \xrightarrow{n\to\infty} 0$

$a(u,u) \geq \alpha \|u\|^2_H$

Proof: Let (u, λ_u) be the solution of (VI_q).

$\Rightarrow \lambda_u u = |u|$

$\Rightarrow \lambda_u^2 u = u$ a.e.

2 cases:
$|\lambda_u| = 1$: $(**)$ ✓
$|\lambda_u| < 1$: $u = 0 \Rightarrow \lambda_u(1+\dots) = \lambda_u$ ✓

$$\boxed{P_\Lambda(\lambda_u + \rho g u)} = P_\Lambda(\lambda_u(\overbrace{1 + \rho g |u|}^{\geq 1})) = P_\Lambda(\lambda_u) \boxed{\underset{|\lambda_u| \leq 1}{=} \lambda_u}$$

$P_\Lambda(cb)$, $|b|, c > 1$
$= P_\Lambda(b)$

Use $\|P_\Lambda b - P_\Lambda a\|_{L^2} \leq \|b - a\|_{L^2}$ (exercise, if you don't see it)

$$\|\lambda^n - \lambda_u\|^2_{L^2(\Gamma_c)} \overset{\text{+def. of } \lambda^n}{=} \|P_\Lambda(\lambda^{n-1} + \rho g u^n) - P_\Lambda(\lambda_u + \rho g u)\|^2_{L^2(\Gamma_c)}$$

$$\leq \|\lambda^{n-1} - \lambda_u + \rho g(u^n - u)\|^2_{L^2(\Gamma_c)}$$

$$= \int_{\Gamma_c} (\lambda^{n-1} - \lambda_u)^2 + \underbrace{\rho^2 \int_{\Gamma_c} g^2 (u^n - u)^2}_{\leq \rho^2 \|g\|^2_{L^\infty} \|u^n - u\|^2_{L^2}} + 2\rho \int_{\Gamma_c} (\lambda^{n-1} - \lambda_u)\cdot(u^n - u) g$$

$$\Rightarrow \boxed{\|\lambda^{n-1} - \lambda_u\|^2_{L^2} - \|\lambda^n - \lambda_u\|^2_{L^2} \geq -\rho^2 \|g\|^2_{L^\infty} \|u^n - u\|^2_{L^2} - 2\rho \int_{\Gamma_c} (\lambda^{n-1} - \lambda_u)(u^n - u) g}$$

(VI_q) w/ $v = u^n$: $\quad a(u, u^n - u) + q(\lambda_u, u^n - u) \geq \langle f, u^n - u \rangle$

$(*)$ w/ $v = u$: $\quad a(u^n, u - u^n) + q(\lambda^{n-1}, u - u^n) \geq \langle f, u - u^n \rangle$ $\quad +$

$$a(u^n - u, u - u^n) + q(\lambda^{n-1} - \lambda_u, u - u^n) \geq 0$$

$$\rightarrow \alpha \|u - u^n\|^2_H \leq a(u - u^n, u - u^n) \leq q(\lambda^{n-1} - \lambda_u, u - u^n)$$

$$= -\int_{\Gamma_c} (\lambda_u - \lambda^{n-1})(u - u^n) g$$

Using that $-\|u^n - u\|_{L^2(\Gamma_c)} \geq -\|u^n - u\|^2_{H^{1/2}(\partial\Omega)} \overset{\text{trace theorem}}{\geq} -C\|u^n - u\|^2_{H^1(\Omega)} \geq -C\alpha^{-1} a(u - u^n, u - u^n)$

$\Rightarrow$ $\boxed{}$ leads to

$$\|\lambda^{n-1} - \lambda_u\|^2_{L^2} - \|\lambda^n - \lambda_u\|^2_{L^2} \geq \rho\,(2 - \rho\,(\alpha^{-1}\|g\|^2_{L^\infty})\, a(u - u^n, u - u^n)$$

If $\rho < 2C\alpha^{-1}\|g\|^{-2}_{L^\infty}$ $\Rightarrow$ > 0

$\rho < 2C\alpha^{-1} \|g\|^{-2}_{L^\infty}$

$$\overset{\text{if } u \neq u^n}{\Rightarrow} \|\sigma^{n+1} - \sigma_u\|^2 > \|\sigma^n - \sigma_u\|^2 \Rightarrow \|\sigma^n - \sigma_u\| \searrow \Rightarrow \|\sigma^n - \sigma_u\| \text{ converges}$$

$$\Rightarrow \|\sigma^{n+1} - \sigma_u\| - \|\sigma^n - \sigma_u\| \to 0$$

$$\Rightarrow \text{From } \boxed{\qquad\qquad} : \ a(u-u^n, u-u^n) \to 0.$$

$\blacksquare$

Di 7.6.2016

Numerical methods

- Active sets: Identify possible contact areas by solving PDE. Shrink possible contact area until convergence
- Uzawa: Iterates projection onto allowed set of Lagrange multipliers.

- Obstacle problem: $E(u) \to \min$ given $u \geq \varphi$ in Ω
- Signiorini problem: $E(u) \to \min$ given $u \geq \varphi$ on $\Gamma_c \subseteq \partial\Omega$
- Friction: $E(u) + \int_{\Gamma_c} g|u| \to \min$

TODAY

- Penalty method: $E(u) + \frac{1}{\varepsilon}(\quad) \longrightarrow \min$ given ~~constraints~~

  $\to \infty$ if constraints are not satisfied

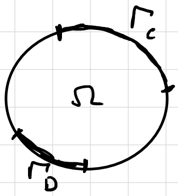
  unconstrained minimization problem, recover solution to (VI) as $\varepsilon \to 0^+$.

- Newton method for $\Phi(u) = 0$, where Φ is not C^1. (but Lipschitz)

Signiorini problem: $E(u) \to \min$ given $u \leq \varphi$ on $\Gamma_c \subseteq \partial\Omega$

$\Leftrightarrow$ (*) Find $u \in K$: $\int_\Omega \nabla u \nabla(v-u) \geq \int_\Omega f(v-u) \quad \forall v \in K$ where $K = \{v \in H^1(\Omega) : v \leq \varphi \text{ on } \Gamma_c, \ v = 0 \text{ on } \Gamma_D\}$

to get a unique solution ($\Rightarrow$ coercivity of $a(u,v) = \int_\Omega \nabla u \nabla v$)



As before, it will be useful to introduce a Lagrange multiplier that quantifies the violation of "=" in (*)

$$\Lambda := \{\lambda \in H^{-1/2}(\Omega) : \underbrace{\int_{\Gamma_c} \lambda v \geq 0 \quad \forall v \in H_0^{1/2}(\Gamma_c), v \leq 0\}}_{\text{think "}\lambda \leq 0\text{"}}$$

$\left(\text{think } \lambda = \frac{\partial u}{\partial n}\Big|_{\Gamma_c}\right)$

(*) is equivalent to the mixed problem:

(**) Find $(u, \lambda) \in H^1_{\Gamma_D}(\Omega) \times \Lambda$ s.t.
$$\left\{\begin{array}{l} a(u,v) - \int_{\Gamma_c} \lambda v = \langle f, v \rangle \quad \forall v \in H^1_{\Gamma_D}(\Omega) \\ \underbrace{\int_{\Gamma_c} (\mu - \lambda)\underbrace{(u - \varphi)}_{\text{this is supposed to be } \leq 0} \geq 0}_{\text{Say } \mu = 0. \text{ Then if } u < \varphi:\ \lambda(u-\varphi) = 0 \text{ a.e.}} \quad \forall \mu \in \Lambda \end{array}\right.$$

$H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$

For the penalty method: Heuristics: [We will minimize something like $E(u) + \frac{1}{2\varepsilon}\int_{\Gamma_c} (u-\varphi)_+^2$ (and get back to (**) soon)

positive part

If we let $\varepsilon \searrow 0$, we hope to get back the exact solution which satisfies $u \leq \varphi$

(PM) Find $u^\varepsilon \in H^1_{\Gamma_D}(\Omega)$: $\quad a(u^\varepsilon, v) + \frac{1}{\varepsilon}\int_{\Gamma_c} (u^\varepsilon - \varphi)_+ v = \langle f, v\rangle \quad \forall v \in H^1_{\Gamma_D}(\Omega)$

We are going to show that $u^\varepsilon \to u$ as $\varepsilon \to 0^+$, where u^ε is the unique solution to (PM).

For a numerical method: (PM_h) FE discretization of PM

Convergence analysis is standard:

$$\|u^\varepsilon - u_h^\varepsilon\|_{H^1} \to 0 \quad h \to 0$$

To understand the numerical approximation of (*) or (**) by (PM_h): $\|u - u_h^\varepsilon\| \leq \|u - u^\varepsilon\| + \|u^\varepsilon - u_h^\varepsilon\|$

Solution of (*) or (**) $\quad (PM_\varepsilon) \quad (PM)$

$\Rightarrow$ Understanding $\|u - u^\varepsilon\|$ is enough also to understand the convergence of the numerical method (PM_h)

We first relate $(u-\varphi)_+$ to λ from (**).

Lemma: Let $u \in H^1_{\Gamma_D}(\Omega)$, $\lambda \in \Lambda$ be a solution of (**) and $p^\varepsilon := \frac{1}{\varepsilon}(u^\varepsilon - \varphi)_+$ from (PM)

$$\Rightarrow \int_{\Gamma_c} (\lambda - p_\varepsilon)(u-\varphi) \geq 0$$

Proof: (**) with $\mu = 0$: $-\int_{\Gamma_c} \lambda(u-\varphi) \geq 0$

(**) with $\mu = 2\lambda \in \Lambda$

$$\int_{\Gamma_c} \lambda(u-\varphi) \geq 0$$

$$\Rightarrow \int_{\Gamma_c} \lambda(u-\varphi) = 0$$

Because (**) is equivalent to (*): $u \in K$

$$\Rightarrow (u-\varphi)_+ = 0$$

$$\Rightarrow \int_{\Gamma_c} (\lambda - p^\varepsilon)(u-\varphi) = -\int p^\varepsilon (u-\varphi) \underset{\text{Def. of } p^\varepsilon}{=} -\int \frac{1}{\varepsilon}(u^\varepsilon - \varphi)_+ \big[\underbrace{(u-\varphi)_+}_{=0} + (u-\varphi)_-\big]$$

≥ 0

∎

Lemma: $\displaystyle\int_{\Gamma_c} (p^\varepsilon - \lambda)(u^\varepsilon - \varphi)_- \leq 0$

Proof: $\displaystyle\int p^\varepsilon (u^\varepsilon - \varphi)_- = \underbrace{\int (u^\varepsilon - \varphi)_+ (u^\varepsilon - \varphi)_-}_{=0} = 0$

$$\Rightarrow \int_{\Gamma_c} (\rho^\varepsilon - \lambda)(u^\varepsilon - \varphi)_- = -\int_{\Gamma_c} \lambda (u^\varepsilon - \varphi)_- \leq 0$$

$\blacksquare$

Compare (PM) and (**): subtract (PM) from (**$_1$)

$$-\left|\begin{array}{l} a(u,v) - \int_{\Gamma_c} \lambda v = \langle f, v\rangle \\ a(u^\varepsilon, v) - \int_{\Gamma_c} \rho^\varepsilon v = \langle f, v\rangle \end{array}\right.$$

$$\Rightarrow a(u-u^\varepsilon, v) = \int_{\Gamma_c} (\lambda - \rho^\varepsilon) v \qquad \forall v$$

$$\Rightarrow a(u-u^\varepsilon, u-u^\varepsilon) = \int_{\Gamma_c} (\lambda - \rho^\varepsilon)(u-u^\varepsilon)$$

$$= \underbrace{\int_{\Gamma_c} (\lambda - \rho^\varepsilon)(u-\varphi)}_{\geq 0} + \int_{\Gamma_c} (\rho^\varepsilon - \lambda)(u^\varepsilon - \varphi)$$

we hope to get eventually (book morning)
$\Rightarrow$ Lemma: $\|\lambda - \rho^\varepsilon\|_{H^{-1/2}} \leq c \|u-u^\varepsilon\|_{H^1(\Omega)}$

Di 7.6.2016 NM

(PM) $\qquad a(u^\varepsilon, v) - \langle \rho^\varepsilon, v\rangle_{\Gamma_c} = \langle f, v\rangle_\Omega \qquad \forall v \in H^1_{\Gamma_D}(\Omega)$

Penalty: $\qquad \rho^\varepsilon := -\frac{1}{\varepsilon}(u^\varepsilon - \varphi)_+ \leq 0$

(**) $\qquad a(u,v) - \langle \lambda, v\rangle_{\Gamma_c} = \langle f, v\rangle_\Omega \qquad \forall v \in H^1_{\Gamma_D}(\Omega)$

Mixed: $\qquad \langle \mu - \lambda, u - \varphi\rangle_{\Gamma_c} \geq 0 \qquad \forall \mu \in \Lambda$

(PM$_h$)
Discretized: $\qquad a(u_h^\varepsilon, v_h) - \langle \rho_h^\varepsilon, v_h\rangle_{\Gamma_c} = \langle f, v_h\rangle \qquad \forall v_h \in H_h$

$$\rho_h^\varepsilon := -\frac{1}{\varepsilon}(u_h^\varepsilon - \varphi)_+$$

$$\boxed{\alpha \|u-u^\varepsilon\|_H^2 \leq a(u-u^\varepsilon, u-u^\varepsilon) \leq \langle \lambda - \rho^\varepsilon, \varepsilon\lambda\rangle \quad \text{algebra! (see copy)}}$$

Summary of this morning

$$\boxed{\text{Lemma: } \|\lambda - \rho^\varepsilon\|_{H^{-\frac{1}{2}}(\Gamma_c)} \leq C \|u-u^\varepsilon\|_{H^1(\Omega)}}$$

Proof: $\|\lambda - \rho^\varepsilon\|_{-\frac{1}{2}} \overset{\text{dual characterization of norm (Hahn-Banach)}}{=} \sup_{v \in H^{1/2}} \frac{\langle \lambda - \rho^\varepsilon, v\rangle}{\|v\|_{1/2}}$

Trace $H^1(\Omega) \to H^{1/2}(\Gamma_c)$ surjective

$$\leq C \cdot \sup_{v \in H^1(\Omega)} \frac{\langle \lambda - \rho^\varepsilon, v|_{\Gamma_c}\rangle}{\|v\|_{H^1(\Omega)}}$$

$$a(u-u_\varepsilon, v) = \langle \lambda - p^\varepsilon, v\rangle_{\Gamma_c} \qquad \forall v$$

$$= C \cdot \sup \frac{a(u-u_\varepsilon, v)}{\|v\|_{H^1(\Omega)}} \leq \tilde{C}\, \|u-u_\varepsilon\|_{H^1}.$$

$\blacksquare$

Theorem: $\|\lambda - p^\varepsilon\|_{H^{\frac{1}{2}}} \leq C \cdot \|u-u^\varepsilon\|_{H^1(\Omega)} \leq \tilde{C} \|\varepsilon\lambda\|_{H^{\frac{1}{2}}(\Gamma_c)}$.

Remark: If $\lambda \in H^{\frac{1}{2}} \Rightarrow \begin{matrix} u \leftarrow u^\varepsilon \\ \lambda \leftarrow p^\varepsilon \end{matrix}$ as $\varepsilon \to 0^+$. Recall $\lambda \in \Lambda \subseteq H^{-\frac{1}{2}}$

Proof: From $\boxed{}$: $\alpha \|u-u^\varepsilon\|^2 \leq \langle \lambda - p^\varepsilon, \varepsilon\lambda\rangle$

$$\leq \|\lambda - p^\varepsilon\|_{-\frac{1}{2}} \|\varepsilon\lambda\|_{\frac{1}{2}}$$

$$\text{Lemma} \leq C\, \|u-\cancel{u^\varepsilon}\|_{H^1(\Omega)} \|\varepsilon\lambda\|_{\frac{1}{2}}$$

$\blacksquare$

Theorem: $\|u^\varepsilon - u_h^\varepsilon\|^2_{H^1(\Omega)} + \varepsilon\|p^\varepsilon - p_h^\varepsilon\|^2_{L^2(\Omega)} \leq C\left(\|u^\varepsilon - w_h\|^2_{H^1(\Omega)} + \varepsilon^{-1}\|w_h - u^\varepsilon\|^2_{L^2(\Gamma_c)}\right) \quad \forall w_h \in H_h$

Céa-Lemma for the discretization of (PM)
(a priori estimate)

The proof will rely on the following lemma:

Lemma: $\|\varepsilon^{1/2}(p^\varepsilon - p_h^\varepsilon)\|^2_{L^2(\Gamma_c)} \leq -\langle p^\varepsilon - p_h^\varepsilon, u^\varepsilon - u_h^\varepsilon\rangle$

Proof of Lemma: $a = u^\varepsilon - \varphi,\ a_+ = -\varepsilon p^\varepsilon \qquad a_+ a_- = b_+ b_- = 0$

$b = u_h^\varepsilon - \varphi,\ b_+ = -\varepsilon p_h^\varepsilon$

Note that:
$$\int_{\Gamma_c} (a_+ - b_+)(u^\varepsilon - u_h^\varepsilon) = (a_+ - b_+)(\overset{a_+ + a_-}{a} - \overset{b_+ + b_-}{b})$$

$$= (a_+ - b_+)^2 + \underbrace{(a_+ - b_+)(a_- - b_-)}_{= -a_+ b_- - b_+ a_- \geq 0}$$

$$\geq (a_+ - b_+)^2$$

$$= \int_{\Gamma_c} \varepsilon^2 (p^\varepsilon - p_h^\varepsilon)^2$$

$\blacksquare$

Proof of the theorem: subtract (PM) with $v = v_h$ from (PM_h):

$$- \left| \begin{array}{l} (PM) \Rightarrow a(u^\varepsilon, v_h) - \langle p^\varepsilon, v_h\rangle = \langle f, v_h\rangle \\ (PM_h) \Rightarrow a(u_h^\varepsilon, v_h) - \langle p_h^\varepsilon, v_h\rangle = \langle f, v_h\rangle \end{array}\right.$$

$$a(u^\varepsilon - u_h^\varepsilon, v_h) - \langle p^\varepsilon - p_h^\varepsilon, v_h\rangle = 0 \qquad \forall v_h \in H_h$$

Let $w = u_h^\varepsilon - v_h$

$$\Rightarrow a(u^\varepsilon - u_h^\varepsilon, u_h^\varepsilon - w_h) - \langle p^\varepsilon - p_h^\varepsilon, u_h^\varepsilon - w_h\rangle = 0$$

An identity: $a(u^\varepsilon - u_h^\varepsilon, u^\varepsilon - u_h^\varepsilon) - \langle p^\varepsilon - p_h^\varepsilon, u^\varepsilon - u_h^\varepsilon\rangle \overset{\text{coercivity + Lemma}}{\geq} \alpha \|u^\varepsilon - u_h^\varepsilon\|_{H^1}^2 + \|\varepsilon^{1/2}(p^\varepsilon - p_h^\varepsilon)\|_{L^2}^2$

$$= a(u^\varepsilon - u_h^\varepsilon, u^\varepsilon - \cancel{u_h^\varepsilon}) - \langle p^\varepsilon - p_h^\varepsilon, u^\varepsilon - \cancel{u_h^\varepsilon}\rangle$$

$$+ a(u^\varepsilon - u_h^\varepsilon, \cancel{u_h^\varepsilon} - w_h) - \langle p^\varepsilon - p_h^\varepsilon, \cancel{u_h^\varepsilon} - w_h\rangle$$

$$= a(u^\varepsilon - u_h^\varepsilon, u^\varepsilon - w_h) - \langle p^\varepsilon - p_h^\varepsilon, u^\varepsilon - w_h\rangle$$

$$\leq C\left(\|u^\varepsilon - u_h^\varepsilon\|_{H^1}\|u^\varepsilon - w_h\|_{H^1} + \|\varepsilon^{1/2}(p^\varepsilon - p_h^\varepsilon)\|\,\|u^\varepsilon - w_h\|/\varepsilon^{1/2}\right)$$

(Young's inequ.) $$\leq \tilde{C}\left(\delta\|u^\varepsilon - u_h^\varepsilon\|^2 + \frac{1}{\delta}\|u^\varepsilon - w_h\|^2 + \delta\|\varepsilon^{1/2}(p^\varepsilon - p_h^\varepsilon)\|_{L^2}^2 + \frac{1}{\delta}\|u^\varepsilon - w_h\|^2/\varepsilon\right)$$

Choose δ small $\Rightarrow$ can move δ^1-terms to the left hand side

∎

Conclusion: $u^\varepsilon \to u$ as $\varepsilon \to 0^+$

$u_h^\varepsilon \to u^\varepsilon$ as $h \to 0^+$

If $\varepsilon \gg h \to 0^+ \Rightarrow u_\varepsilon^h \to u$ $\quad \varepsilon^{-1}h \to 0$ e.g. $\varepsilon = h^{1-\delta}$, $\delta \in (0,1)$

if we use pw. linear elements

(4) Semismooth Newton methods

"Newton's method for functions with kinks"

Recall for Signiorini problem:

$\boxed{u - \varphi \leq 0,\ \lambda \leq 0 \qquad \lambda \cdot (u - \varphi) = 0}$ this is actually one of the ways we stated the Signiorini problem at the very beginning

Discretized (mixed) problem: $\quad a(u_h, v_h) - \langle \lambda_h, v_h\rangle_{\Gamma_C} = \langle f, v_h\rangle_\Omega \quad \forall v_h \in H_h$

$$\langle \mu_h - \lambda_h, u_h - \varphi\rangle \geq 0 \quad \forall \mu_h \in \Lambda$$

Choose $\mu_h = 2\lambda_h$: $\langle \lambda_h, u_h - \varphi\rangle \geq 0$

$\mu_h = 0$: $\langle -\lambda_h, u_h - \varphi\rangle \geq 0$

$\Rightarrow \langle \lambda_h, u_h - \varphi\rangle = 0$ Also $\lambda_h \leq 0$, $u_h - \varphi \leq 0$ if $\overset{\Lambda_h \subseteq \Lambda}{H_h \subseteq H^1(\Omega)}$

$\Rightarrow$ relations $\boxed{}$ also hold for discretized problem

Discrete problem $\boxed{\begin{array}{l} A u_h - \lambda_h = F \\ \lambda_h \leq 0,\ u_h - \varphi \leq 0,\ \lambda_h \cdot (u_h - \varphi) = 0 \end{array}}$ ⟵ "complementarity conditions"

Rewrite as nonlinear <u>equation.</u> Solve with (semismooth) Newton method.

Abstract problem: $\begin{cases} \mathbb{F}(u,\lambda)=0 \\ \lambda \geq 0, \quad u \geq 0, \quad \lambda \cdot u = 0 \end{cases}$ $\qquad \lambda = -\lambda_h \quad u = (u_h - \varphi)$ $\qquad \mathbb{F}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$

$$\Leftrightarrow \begin{cases} \mathbb{F}(u,\lambda) = 0 \\ \Phi(u,\lambda) = 0 \end{cases}$$

<u>Exercise:</u> $\lambda \geq 0, u \geq 0, u \cdot \lambda = 0 \iff \Phi(u,\lambda) = 0$ where

1.) $\Phi_0(u,\lambda) = u + \lambda - \sqrt{u^2 + \lambda^2}$ Fischer–Burmeister nonlinear complementarity function

2.) $\Phi_1(u,\lambda) = \max\{0,u\} \cdot \max\{u,0\}$

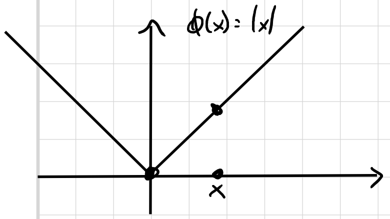
3.) $\Phi_\mu(u,\lambda) = (1-\mu)\Phi_0 + \mu\Phi_1 \qquad \mu \in (0,1)$

$\Phi_0, \Phi_1, \Phi_\mu \notin C^1$, but Lip.

Let $\Phi: \mathbb{R}^N \to \mathbb{R}^m$

$\boxed{\text{Theorem: (Rademacher) If } \Phi \text{ Lipschitz} \Rightarrow \text{For almost every } x: \Phi \text{ is differentiable at}}$

<u>Definition:</u> The Clark subdifferential of Φ at x is the set $\partial\Phi(x) = \text{conv}\,\{H : \exists \text{ sequence } (x_k) \text{ of points, where } \Phi' \text{ exists s.t. } x_k \to x \text{ and } \Phi'(x_k) \to H\}$



$\partial\Phi(x) = \{1\} \quad, x > 0$

$\partial\Phi(x) = \{-1\} \quad, x < 0$

$\partial\Phi(0) = \text{conv}\{+1,-1\} \overset{!}{=} [-1,1]$