Conclusion: (TP) is elliptic in the sense of Shapiro-Lopatinski

$\iff \alpha \neq -1$

Theorem (Well posedness):

Assume $(*)$ is elliptic in a bounded smooth domain Ω.

Then the space of solutions to $(*)$ is at most finite-dim. It is nonempty for f, g outside finite-dimensional subspace.

Also: $\|u\|_{H^s(\Omega)} \leq C \; \|f\|_{H^{s-2}} + \|g\|_{H^{s-\mu}} + \|u\|_{H^{s-1}}$ for $s > \mu$, where $\mu = \frac{1}{2}$ (Dirichlet), $\frac{3}{2}$ in general

III General results on variational inequalities:

The Theorems of Lions-Stampacchia and Glowinski

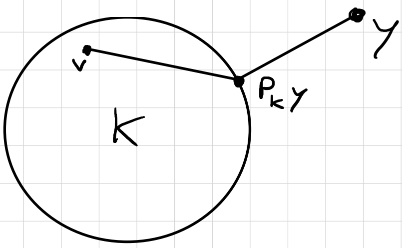
a continuous, coercive, bilinear form on H

$f \in H^*$

$\emptyset \neq K \subseteq H$ closed convex

As a is not assumed to be symmetric, the variational inequality might not come from an energy-minimization problem any longer.

(VI) Find $u \in K$: $a(u, v-u) \geq \langle f, v-u \rangle \quad \forall v \in K$



Theorem (Lions-Stampacchia):

(VI) has a unique solution.

Challenge: The map $y \mapsto$ closest point $P_K y$ to y in K is not linear

This morning: $P_K y$ is the unique point in K s.t. $\forall v \in K : (y - P_K y, v - P_K y) \leq 0$

Proof: Uniqueness: Let u_1, u_2 be solutions to (VI)

$$+\left| \begin{array}{ll} a(u_1, u_2 - u_1) \geq \langle f, u_2 - u_1 \rangle & \forall v \in K \\ a(u_2, u_1 - u_2) \geq \langle f, u_1 - u_2 \rangle & \end{array} \right.$$

$\Rightarrow -a(u_1 - u_2, u_1 - u_2) = a(u_1 - u_2, u_2 - u_1) \geq 0$

$-a(u_1 - u_2, u_1 - u_2) \leq -\alpha \|u_1 - u_2\|_H^2$ by coercivity $\Rightarrow \|u_1 - u_2\|_H \leq 0 \Rightarrow u_1 = u_2$ $\blacksquare$

Existence: $a(u,v) \overset{\rho > 0}{\Rightarrow} \left(\rho(Au - f) \overset{+u-u}{,} v - u \right) \geq 0$ (VI)

$$\Leftrightarrow (\underbrace{u-\varrho(Au-f)}_{y}-\underset{P_K y}{\underset{\shortparallel}{u}}, v-\underset{P_K y}{\underset{\shortparallel}{u}}) \le 0 \quad \forall v\in K \quad \forall \varrho>0$$

This is equivalent to $\quad u = P_K(u-\varrho(Au-f))$.

Let's study $W_\varrho(v) := P_K(v-\varrho(Av-f))$.

$$W_\varrho : H \to K \subset H$$

$\forall v_1, v_2 \in H$: $\|W_\varrho(v_1)-W_\varrho(v_2)\|_H^2 = \|P_K(v_1-\varrho(Av_1-f)) - P_K(v_2-\varrho(Av_2-f))\|_H^2$

$\leftarrow \|P_K y_1 - P_K y_2\| \le \|y_1-y_2\|$

$\le \|v_1-v_2-\varrho A(v_1-v_2)\|_H^2$

$\le \|v_1-v_2\|_H^2 + \varrho^2\|A(v_1-v_2)\|^2 \underbrace{- 2\varrho a(v_2-v_1, v_2-v_1)}_{\le -2\varrho\alpha\|v_2-v_1\|_H^2}$

$\le (1-2\varrho\alpha + \varrho^2\|A\|^2)\|v_2-v_1\|_H^2 \qquad 1-2\varrho\alpha+\varrho^2\|A\|^2 < 1$ if ϱ is small

$0<\varrho<\frac{2\alpha}{\|A\|^2}$

$\Rightarrow \|W_\varrho(v_1)-W_\varrho(v_2)\|^2 < c^2\|v_1-v_2\|^2$ for a $c<1$ if ϱ small.

$\Rightarrow$ contraction $\Rightarrow$ Banach $\Rightarrow$ unique solution

$\llcorner$ step above not necessary

$\blacksquare$

Approximation of the solution:

Assumptions: $H_h \subset H$ closed subspace (e.g. fin.-dim.)

$K_h \subset H_h$ closed convex sets

(We don't require $K_h \subset K$!)

s.t.: 1.) We only consider sequences $\{v_h\}_h \subseteq H$ which are bounded in H and $v_h \in K_h$ $\forall h$ and $v_h \rightharpoonup v^*$ in K

2.) $\exists \mathcal{X} \subset H$ with $\overline{\mathcal{X}} = K$ and $\exists r_h : \mathcal{X} \to K_h$ s.t. $\lim_{h\to 0} r_h v = v$ strongly in H whenever $v\in\mathcal{X}$.

Approximation problem:

Find $u_h \in K_h$: $a(u_h, v_h-u_h) \ge \langle f, v_h-u_h\rangle \quad \forall v_h\in K_h \qquad (VI_h)$

Theorem (VI_h): admits a unique solution

Proof: Clear from Lions-Stampacchia $\blacksquare$

Theorem (Glowinski):

Assume the above Assumption holds for K and K_h

Then $\lim_{h\to 0^+} \|u_h - u\|_H = 0$.

I.e. the approximation scheme converges

Proof: **3 steps:** ① a priori estimate on u_h

② u_h converges weakly as $h \to 0$

③ u_h converges strongly

Step ①: Show: $\exists c_1, c_2 : \|u_h\|^2 \le c_1 \|u_h\| + c_2$

$(\Rightarrow \|u_h\| \le M$ for some $M(c_1,c_2))$

$$(VI_h) \qquad a(u_h, v_h - u_h) \geq \langle f, v_h - u_h\rangle \quad \forall v_h \in K_h$$

$$\Rightarrow \underset{\alpha\|u_h\|^2}{\overset{\ge}{a(u_h,u_h)}} \le -\langle f, v_h - u_h\rangle + \underbrace{a(u_h, v_h)}_{\langle Au_h, v_h\rangle}$$

$$\le \|f\|(\|v_h\| + \|u_h\|) + \|A\|\|u_h\|\|v_h\|$$

Fix $v_0 \in \mathcal{X}$ and let $v_h = r_h v_0 \in K_h$. From Assumption, part 2, $r_h v_h \to v_0$

$\Rightarrow \|v_h\| \le \tilde{M}$ for some $\tilde{M}$ indep. of h

$$\Rightarrow \alpha\|u_h\|^2 \le \underbrace{\|f\|(\tilde{M}}_{\tilde{c}_2} + \|u_h\|) + \|A\|\,\tilde{M}\,\|u_h\|$$

$\|f\| + \|A\|\,\tilde{M} = \tilde{c}_1$

$$\le \tilde{c}_1 \|u_h\|^2 + \tilde{c}_2 \qquad , \quad c_1 = \frac{\tilde{c}_1}{\alpha}$$

$$c_2 = \frac{\tilde{c}_2}{\alpha}$$

Step ②: Since $\|u_h\|_H \le M$ and the ball $\{x \in H : \|x\| \le M\}$ is compact in the weak topology of H

$\Rightarrow \exists$ convergent subsequence of $\underset{K_h}{\underset{\cap}{u_h}} \rightharpoonup \underset{H}{\underset{\cap}{u^*}}$ wlog $u_h \rightharpoonup u^*$

By Assumption, part 1: $u^* \in K$

Show: u^* is a solution to (VI):

$$a(u_h, v_h - u_h) \ge \langle f, v_h - u_h\rangle$$

Let $v \in \mathcal{X} \subseteq K$, $v_h = r_h v \in K_h$

$$\underbrace{a(u_h,u_h)}_{\xrightarrow[h\to 0]{} u^*} \leq -\langle f, \underset{r_h v \to v}{\underset{\parallel}{v_h}} - u_h\rangle + a(u_h, \underset{r_h v\to v}{\underset{\parallel}{v_h}})$$

$$\to -\langle f, v-u^*\rangle + a(u^*,v)$$

$$\underline{\lim}\, a(u_h,u_h) \leq a(u^*,v) - \langle f, v-u^*\rangle \qquad \forall v\in \mathcal{X}$$

$$\Rightarrow 0 \leq a(u_h-u^*, u_h-u^*) \leq a(u_h,u_h) - a(u_h,u^*) - a(u^*,u_h) + a(u^*,u^*)$$

$$a(u_h,u^*) + a(u^*,u_h) - a(u^*,u^*) \leq a(u_h,u_h)$$

$$\underline{\lim}_{h\to 0} \quad \underset{a(u^*,u^*)}{\downarrow} \qquad \underset{a(u^*,u^*)}{\downarrow}$$

$$\Rightarrow \quad a(u^*,u^*) \leq \underline{\lim}\, a(u_h,u_h) \leq a(u^*,v) - \langle f, v-u^*\rangle$$

$$\Rightarrow \quad a(u^*, u-u^*) \geq \langle f, v-u^*\rangle$$

$\Rightarrow$ u^* satisfies (VI)

As $\overline{\mathcal{X}} = K$, a, f cont., this holds $\forall v\in K$, not just $v\in\mathcal{X}$.

<u>Step 3</u>: $(**)$ w/ $v_h = r_h v$, $v\in\mathcal{X}$

$$a(u_h,u_h) \leq a(u_h, r_h v) - \langle f, r_h v - u_h\rangle$$

Coercivity of a: $a(u_h-u, u_h-u) \geq \alpha \overline{\lim}_{h\to 0} \|u_h-u\|_H^2$

$$\overline{\lim}_{h\to 0}\, a(u_h,u_h) \underbrace{+a(u,u) - a(u_h,u) - a(u,u_h)}_{\xrightarrow[h\to 0]{} -a(u,u)} =$$

u_h converges weakly to u $\Rightarrow$ $a(u_h,u) \to a(u,u)$

$a(u,u_h) \to a(u,u)$

$$a(u_h,u_h) \leq \underbrace{a(u_h, r_h v)}_{= a(u_h, r_h v - v) + a(u_h,v),\ a(u_h, r_h v - v)\to 0} - \langle f, r_h v - u_h\rangle \xrightarrow[h\to 0]{} a(u,v) - \langle f, v-u\rangle$$

$$(***) \qquad \alpha \overline{\lim}_{h\to 0} \|u-u_h\|^2 \leq a(u,v) - \langle f, v-u\rangle - a(u,u)$$

By continuity, $(***)$ extends to all $v\in K$.

Set $v=u$ $\Rightarrow$ $\overline{\lim}_{h\to 0} \|u-u_h\|^2 \leq 0$

∎

So far, we have always considered the variational inequality $(*)$ or (if a is symmetric) equivalently

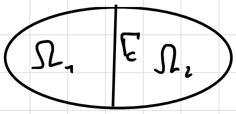
$E(u) = \frac{1}{2} a(u,u) - \langle f, u\rangle \to \min_K$. This proofs existence and uniqueness etc. for

Examples a) – c) from day 1:
- transmission problems (PDEs)
- obstacle problems ($u\geq\psi$ in Ω)
- Signorini problems ($u\geq\psi$ on Γ_c)

← nonlinear term added

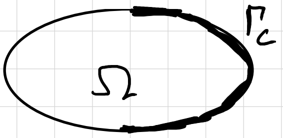
In example d), we had $E(u) = \frac{1}{2} a(u,u) - \langle f,u\rangle + j(u)$, where $a(u,v) = \frac{1}{2}\int_{\Omega_1} (\nabla u_1)(\nabla v_1) + \frac{1}{2}\int_{\Omega_2} (\nabla u_2)(\nabla v_2)$

$\langle f,u\rangle = \int_{\Omega_1} f_1 u_1 + \int_{\Omega_2} f_2 u_2$ and $j(u) = \int_{\Gamma_C} g \|u_1 - u_2\|$, $H = \{(u_1,u_2) \in H^1(\Omega_1)\times H^1(\Omega_2),\ u_1 = u_2 = 0 \text{ on } \partial\Omega\}$



$0 \le g \in L^\infty$

For simplicity of notation, we just prove existence for



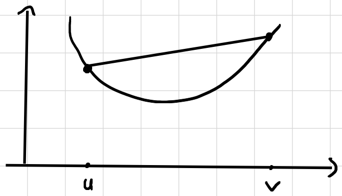
$a(u,v) = \frac{1}{2}\int_\Omega (\nabla u)(\nabla v)$, $\langle f,u\rangle = \int_\Omega fu$, $j(u) = \int_{\Gamma_C} g|u|$ on $H = \{u \in H^1(\Omega): u = 0 \text{ on } \partial\Omega\setminus\Gamma_C\}$

Tresca friction

$$E(u) = \frac{1}{2} a(u,u) - \langle f,u\rangle$$

$$E'(u) = E(u) + j(u)$$

<u>Note:</u>
- $E'(u)$ is coercive in the sense that $E'(u) \to \infty$ as $\|u\|_H \to \infty$.
- $E'(u)$ is convex: $E'(u + \lambda(v-u)) \le (1-\lambda)E'(u) + \lambda E'(v) \quad \forall \lambda \in [0,1]$



$$\Leftrightarrow \boxed{E'(u+\lambda(v-u)) - E'(u) \le \lambda(E'(v) - E'(u))}$$

- $j(v) - j(u) = \int_{\Gamma_C} g(|v| - |u|) \geq \underbrace{\int_{\Gamma_C} g\, \mathrm{sign}(u)\,(v-u)}_{=:\, q_u(v-u)}$, where $|u| = u\cdot \mathrm{sign}(u)$

From $\boxed{}$: $E'(v) - E'(u) \ge \frac{1}{\lambda}\Big(\frac{1}{2}a(u+\lambda(v-u), u+\lambda(v-u)) - \langle f, u+\lambda(v-u)\rangle + j(u+\lambda(v-u)) - \frac{1}{2}a(u,u) + \langle f,u\rangle - j(u)\Big)$

$$= \frac{1}{\lambda}\Big(a(u,\lambda(v-u)) + \frac{1}{2}a(\lambda(v-u),\lambda(v-u)) - \langle f, \lambda(v-u)\rangle + \underbrace{j(u+\lambda(v-u)) - j(u)}_{\ge q_u(\lambda(v-u))}\Big)$$

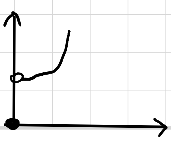
$$\ge a(u,v-u) + \frac{1}{2}a(v-u,v-u) - \langle f, v-u\rangle + q_u(v-u)$$

$$\xrightarrow{\lambda\to 0} a(u,v-u) - \langle f, v-u\rangle + q_u(v-u)$$

$\Rightarrow E'(v) - E'(u) \ge a(u,v-u) - \langle f,v-u\rangle + q_u(v-u)$ (*)

<u>Lemma:</u> $E'(u)$ is weakly lower semicontinuous on H, i.e. for all $\{u_k\} \subseteq H$ s.t. $u_k \rightharpoonup u$ weakly

$$\liminf_{k\to\infty} E'(u_k) \ge E'(u)$$



<u>Proof:</u> From (*) with $v = u_k$ $\Rightarrow E'(u_k) - E'(u) \ge a(u,u_k-u) - \langle f, u_k - u\rangle + q_u(u_k-u)$

$$\xrightarrow{k\to\infty} a(u,u-u) - \langle f, u-u\rangle + q_u(u-u) = 0$$

$$\Rightarrow \liminf E'(u_k) - E'(u) \ge 0$$

∎

Theorem: There exists a minimizer of E'.

Proof: coercivity: $E'(v) \to \infty$ as $v \to \infty$

This implies that $\exists M: \forall \|u\| > M: \quad E'(v) > \inf E' + 1$

Consider $A = \{v \in H: \|v\|_H \leq M\}$. Any minimizer will lie in A: $\inf\limits_{v\in H} E'(v) = \inf\limits_{v\in A} E'(v)$

Also $|E'(v)| \leq C\|v\|_H^2 + D\|v\|_H \leq CM^2 + DM =: \overline{M}$

$\Rightarrow -\overline{M} \leq \inf E'(v) \leq \overline{M}$

Let $\{u_k\} \subseteq A$ be an infimizing sequence: $\inf\limits_{v\in A} E'(v) = \lim\limits_{k\to\infty} E'(u_k)$

By weak compactness of A (Banach-Alaoglu) $\exists$ weakly convergent subsequence u_{k_i} of $u_k \rightharpoonup u \in A$

$$\inf E' = \lim_{i\to\infty} E'(u_{k_i}) \underset{\text{Lemma}}{\geq} E'(u) \geq \inf E'$$

$$\Rightarrow E'(u) = \inf E'$$

$\blacksquare$

Conclusion: the friction problem also admits a solution

Uniqueness: exercise: use the "variational inequality" $a(u, v-u) + j(v) - j(u) \geq \langle f, v-u\rangle \quad \forall v$
(This characterizes the solution u)

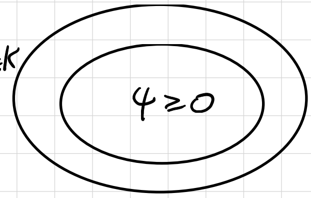
IV Numerical discretization

1.) Obstacle problem:

$$E(u) = \frac{1}{2}\int_\Omega (\nabla u)^2 + \int_\Omega fu \longrightarrow \min_K$$

, $K = \{u \in H_0^1(\Omega): u \geq \psi\}$ where $\psi \in C^2(\Omega)$ and $\psi < 0$ on $\partial\Omega$

Variational inequality (VI): Find $u \in K$ s.t. $\int_\Omega \nabla u \nabla(v-u) \geq \int_\Omega f(v-u) \quad \forall v \in K$



$\Omega \subseteq \mathbb{R}^2$

We introduce a "Lagrange multiplier" $\lambda \in M_+ := \{\lambda \in L^2(\Omega): \lambda \geq 0\}$

This results in the **Variational Equation (VE):** Find $(u,\lambda) \in K \times M_+$ s.t. $\int_\Omega \nabla u \nabla v - \int_\Omega \lambda v \, dx = \int_\Omega fv \quad \forall v \in K$

"$\lambda = Au - f$" violation of $Au = f$

Lemma: If $u \in H^2(\Omega)$ (and one can prove that for $f \in L^2$ it is)

$\Rightarrow \exists! \, (u,\lambda)$ satisfying (VE), where u is the solution to (VI) and $\lambda \in M_+$

s.t. ($\lambda \geq 0$ and $u \geq \psi$ and) $\int_\Omega \lambda(u-\psi) = 0$. ($\Leftrightarrow$ For a.e. $x \in \Omega: u(x) > \psi(x) \Rightarrow \lambda(x) = 0$)

Discretized

Active set algorithm: Fix parameter $c > 0$.

$$A_c := \{x \in \Omega : \lambda(x) - c(u(x) - \psi(x)) > 0\} \quad (\text{for } x \in A_c \text{, either } \lambda(x) > 0 \text{ or } u(x) < \psi(x)\text{?})$$

$$I_c := \{x \in \Omega : \lambda(x) - c(u(x) - \psi(x)) \leq 0\}$$

$\Rightarrow$ Reformulated problem: Find $u \in H_0^1(\Omega)$: $\int_\Omega \nabla u \nabla v - \int_\Omega \lambda v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega) \qquad (*)$

- Initialize $A_{c,h}^{(0)}$ arbitrary

For $n = 1, 2, 3, \dots$

- Find $u^{(n)} \in H_0^1(\Omega)$, $\lambda^{(n)} \in L^2(\Omega)$ — $(\vec{u}_h, \vec{\lambda}_h)$ solving $(*_h)$ with $A_{c,h}^{(n-1)}$

$$\int_\Omega \nabla u^{(n)} \nabla v - \int_\Omega \lambda^{(n)} v = \int f v \qquad \forall v$$

s.t. $u^{(n)} = \psi$ on $A_c^{(n-1)}$, $\lambda^{(n)} = 0$ on $I_c^{(n-1)} = \Omega \setminus A_c^{(n-1)}$

- Update $A_{c,h}^{(n)} := \{x \in \Omega : \lambda^{(n)} - c(u^{(n)}(x) - \psi(x)) > 0\}$ — $p^i : (\lambda_h^{(n)})_i - c((u_h^{(n)})_i - (\psi)_i)$

$$I_c^{(n)} := \Omega \setminus A_c^{(n)}$$

- If $|A_{c,h}^{(n-1)} \setminus A_{c,h}^{(n)}| = 0$ stop

Discretization: We triangulate $\Omega = \bigcup_{T \in \mathcal{T}_h} T$.

$H_h = \{\text{piecewise linear polynomials, continuous} + 0 \text{ at } \partial\Omega\}$

$\overset{!}{=} \operatorname{span}\{\phi^1, \dots, \phi^n\}$

nodes p^j, $\phi^i(p^j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

We consider the discretized minimization problem $E(u) \to \min_{\{v \in H_h : v(p^j) \geq \psi(p^j)\} =: K_h}$

If ψ is piecewise linear, this is conforming $(K_h \subseteq K)$, but in general it is not.

Define vectors: $(\vec{\psi})_i = \psi(p^i)$, $u_h = \sum_i (\vec{u}_h)_i \phi^i$

Discrete Non penetration cond: $(\vec{u})_i \geq (\vec{\psi})_i \quad \forall i$

Stiffness matrix $A_{ij} = a(\phi^i, \phi^j) = \int_\Omega \nabla\phi^i \nabla\phi^j$

Load vector $(\vec{f})_i = \int_\Omega f \phi^i$

Lagrange mult. $(\vec{\lambda}_h)^i \geq 0$

$\Rightarrow$ $\boxed{\text{Discretized version of } (*) \ (*_h) : \text{Find } \vec{u}_h \text{ and } \vec{\lambda}_h \text{ s.t. } A\vec{u}_h - \vec{\lambda}_h = \vec{f}, \ (\vec{u}_h)_i = (\vec{\psi})_i \text{ for } p^i \in A_c, \ (\vec{\lambda}_h)_i = 0 \text{ for } p^i \notin A_c}$

Definition: A matrix A is called an M-matrix provided that $A = s \cdot \mathbb{1} - B$, where $s \geq$ largest eigenvalue of B and $B_{ij} \geq 0 \ \forall i,j$.

Assume that $A_{c,h}^{(0)}$ is large enough.

Lemma: Assume $\mathcal{T}_h$ is such that the stiffness matrix is an M-matrix. Then the discretized active set algorithm stops after a finite number of steps and returns the solution $(\vec{u}_h, \vec{\lambda})$ to $(*_h)$.

Moreover $\quad \vec{\psi} \le u_h^{(1)} \le u_h^{(2)} \le \ldots \le \vec{u}_h$

$\mathcal{T}_h \ge A_{c,h}^{(1)} \ge A_{c,h}^{(2)} \ge \ldots \ge A_{c,h}^{(*)}$

Proof: $\vec{\delta}_u^{(n-1)} := \vec{u}_h^{(n)} - \vec{u}_h^{(n-1)}$

$\vec{\delta}_\lambda^{(n-1)} := \vec{\lambda}_h^{(n)} - \vec{\lambda}_h^{(n-1)}$

$$\Rightarrow A\delta_u^{(n)} = \delta_\lambda^{(n)} \qquad (**)$$

$$A u_h^{(n)} - \lambda_h^{(n)} = f = A\vec{u}_h^{(n-1)} - \vec{\lambda}_h^{(n-1)}$$

$$\Rightarrow (**)$$

Permute the P^i s.t. active nodes are first.

$P_{(n)}$ permutation matrix.

$$\Rightarrow \quad P_{(n)} A P_{(n)}^T = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$$

$$\delta_u^{(n)} = (\delta_u^{(n)+}, \delta_u^{(n)-}), \quad \delta_\lambda^{(n)} = (\delta_\lambda^{(n)+}, \delta_\lambda^{(n)-})$$

$$\begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} \begin{pmatrix} \delta_u^{(n)+} \\ \delta_u^{(n)-} \end{pmatrix} = \begin{pmatrix} \delta_\lambda^{(n)+} \\ \delta_\lambda^{(n)-} \end{pmatrix}$$

$$A_{--}\delta_u^{(n)-} = -A_{-+}\delta_u^{(n)+} + \delta_\lambda^{(n)-} \Rightarrow \delta_u^{(n)-} = -A_{--}^{-1}A_{-+}\delta_u^{(n)+} + A_{--}^{-1}\delta_\lambda^{(n)-} \ge 0$$

Why "≥ 0"? $\quad \delta_\lambda^{(n)-} \ge 0$ because $\lambda^{(n)} = 0$ on the inactive set

$\delta_u^{(n)+} \ge 0$ because $u = \psi$ on active set

$\Rightarrow A_{--}^{-1}\delta_\lambda^{(n)-} \ge 0$ and $A_{-+}\delta_u^{(n)+} \le 0$ $\quad$ because A is an M-matrix

$$\Rightarrow A_{--}\delta_u^{(n)-} \ge 0. \Rightarrow \delta_u^{(n)-} \ge 0 \Rightarrow u_i^{(n+1)} \ge u_i^{(n)} \quad \text{for } i \notin A_c$$

$\Rightarrow \lambda - c(u-\psi) \searrow$ on inactive set $\Rightarrow$ points in inactive set stay inactive.

Therefore either $A_c^{(n)} = A_c^{(n+1)}$ ($\Rightarrow$ algorithm stops)

or $A_c^{(n)} \supsetneq A_c^{(n+1)}$ (can happen only (#vertices) times)

∎

Exercise: If $A_c^{(0)}$ = all nodes $\Rightarrow$ the algorithm returns a solution to $(*_h)$

Remaining Question: When is A an M-matrix?

Lemma: Let the interior angles of all $T \in \mathcal{T}_h$ be $< \frac{\pi}{2}$

$$\Longrightarrow A \text{ is an } M\text{-matrix}$$