Freitag keine VO: $\Rightarrow$ Mo: 13–14:30
Mi: 15:30 – 17

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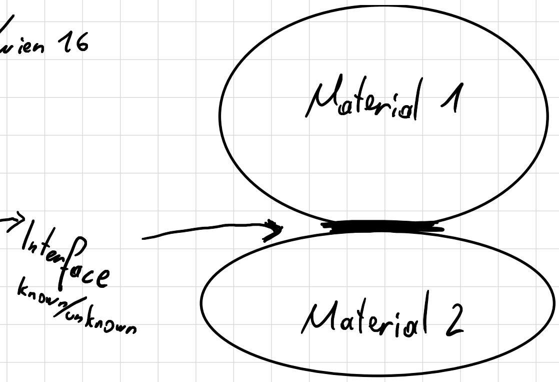


- conservation of momentum + displacement $\Rightarrow$ standard boundary value (transmission) problems
- friction, opening of gaps, ...

These are boundary value problems involving inequalities

$\boxed{\text{Topics in } 1^{st} \text{ week:}}$

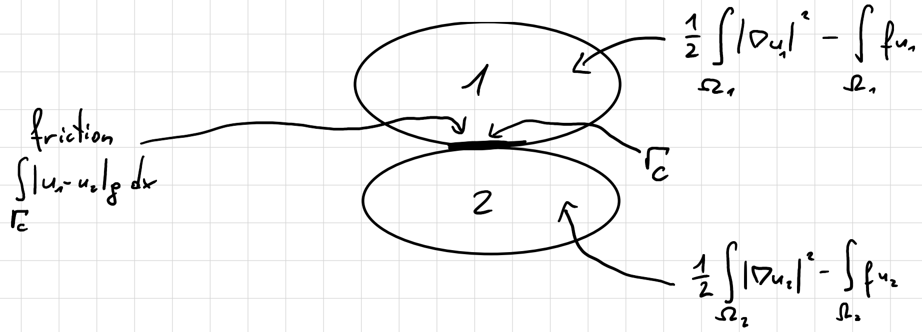
- What are the equations?
- Well-posedness $\leftarrow$ Variational inequalities
- Numerical approximation w/ FEM+BEM
- Error analysis and adaptive mesh refinements
- Solvers

$\boxed{2^{nd} \text{ week:}}$

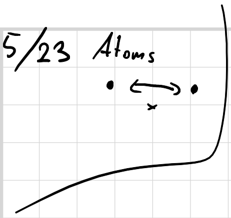
- advanced discretisations
  penalty, mixed or hp, ...
- FEM-BEM coupling
- time-domain
- ...
- (less advanced topics)

slide 2/7

Total Energy = Energy of 1 + Energy of 2 + Interaction between 1/2



5/23 Atoms



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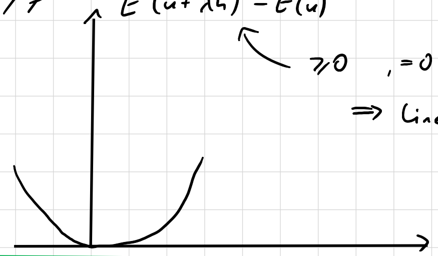


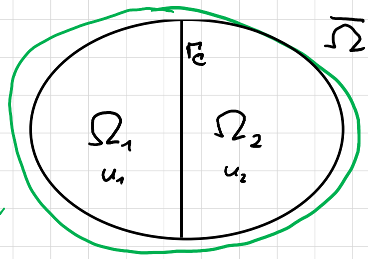
$E(u+\lambda h) - E(u)$

≥ 0 , $=0$ at $\lambda = 0$

$\Rightarrow$ Linear term $\equiv 0$

I Example problems and basic analysis

a)



$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$

Transmission problems: $E(u_1, u_2) = \frac{1}{2} \int_{\Omega_1} (\nabla u_1)^2 dx + \frac{\alpha}{2} \int_{\Omega_2} (\nabla u_2)^2 dx$

$$- \int_{\Omega_1} f_1 u_1 \, dx - \int_{\Omega_2} f_2 u_2 \, dx$$

solidly glued together: $u_1 = u_2$ on Γ_c

say Dirichlet conditions on $\partial\Omega$: $u_{1,2} \in H^1(\Omega_{1,2})$, $u_{1,2}|_{\partial\Omega} = 0$

u_1 lives on Ω_1

u_2 — " — Ω_2

$u_{1,2}$ is 0 on $\Omega_{1,2}$, $u_{2,1}$ might not

Minimize $E(u_1, u_2)$ over all such u_1, u_2

$E(u_1 + \lambda h_1, u_2 + \lambda h_2) - E(u_1, u_2)$ $\qquad h_1 = h_2$ on Γ_c

$$= \ldots = -\lambda \int_{\Omega_1} (\Delta u_1 + f_1) h_1 - \lambda \int_{\Omega_2} (\alpha \Delta u_2 + f_2) h_2$$

$$-\lambda \int_{\Gamma_c} \underbrace{\left((\vec{n} \cdot \nabla u_1) h_1 - \alpha (\vec{n} \nabla u_2) h_2 \right)}_{= (\vec{n} \cdot \nabla u_1 - \alpha \vec{n} \cdot \nabla u_2) h_1} ds + \mathcal{O}(\lambda^2)$$

normal of Ω_1 is equal to $-$normal of Ω_2

As before, when $h_1 = h_2 = 0$ on Γ_c $\Rightarrow -\Delta u_1 = f_1$ in Ω_1

$$-\alpha \Delta u_2 = f_2 \quad \text{in } \Omega_2$$

Then also the second row has to be $\equiv 0$:

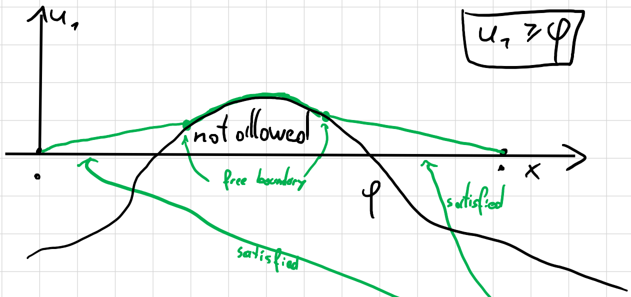
$$-\lambda \int_{\Gamma_c} \left((\vec{n} \nabla u_1) - \alpha (\vec{n} \cdot \nabla u_2) \right) h_1 \, ds = 0 \qquad \forall h_1$$

$$\Rightarrow (\ \ldots \ ) = 0, \text{ i.e. } \frac{\partial u}{\partial n} - \alpha \frac{\partial u_2}{\partial n} = 0$$

To summarize: Minimizing $E(u_1,u_2)$ implies the PDE
(exercise: is equivalent to)

$$\boxed{\begin{aligned} -\Delta u_1 &= f_1 \quad \text{in } \Omega_1 \\ -\alpha\Delta u_2 &= f_2 \quad \text{in } \Omega_2 \quad \leftarrow \text{continuity of displacement} \\ u_1 - u_2 &= 0 \quad \text{on } \Gamma_C \quad \leftarrow \text{continuity of normal stress} \\ \frac{\partial u_1}{\partial n} - \alpha\frac{\partial u_2}{\partial n} &= 0 \quad \text{on } \Gamma_C \\ u_1 = 0 &\text{ resp. } u_2 \text{ on } \partial\Omega \end{aligned}}$$

b) Obstacle problems: Ω_2 is hard/cannot be penetrated.



$$E(u_1) = \frac{1}{2}\int_{\Omega_1} (\nabla u_1)^2 - \int_{\Omega_1} f u_1$$

minimize over all $u_1 \in H_0^1$ that satisfy $u_1 \geq \varphi$ for $\varphi \in C^\infty$ given.

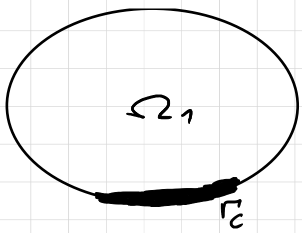
+ Dirichlet bc.

$\Rightarrow$ As long as $u_1 > \varphi \Rightarrow$ $\boxed{\begin{aligned} -\Delta u_1 &= f \quad \text{in } \{u_1 > \varphi\} \leftarrow \text{Problem: not known} \\ u_1 &= \varphi \quad \text{else} \end{aligned}}$

Free boundary problem: You don't know what $\{u_1 = \varphi\}$ is.

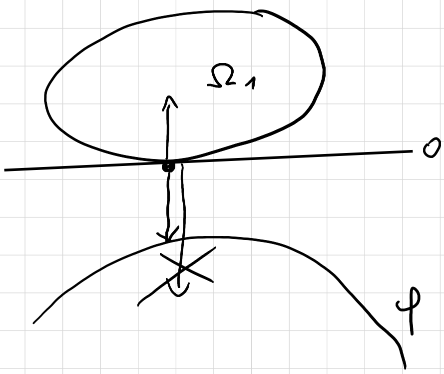
One can show that u_1 belongs to $C^{1,1}(\Omega_1)$

c) Signorini problem: $E(u_1) = \frac{1}{2}\int_{\Omega_1} (\nabla u_1)^2 - \int_{\Omega_1} f u_1$, $u_1 \geq \varphi$ on $\Gamma_C \subseteq \partial\Omega$
(Non penetration) Car crash $\perp$ wall



$\longrightarrow \min_{\substack{u_1 \geq \varphi \text{ on } \Gamma_C \\ u_1 \in H^1(\Omega_1)}}$

$$\Longleftrightarrow \begin{cases} -\Delta u_1 = f & \text{in } \Omega_1 \\ \frac{\partial u_1}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_C \\ u_1 \geq \varphi,\ \frac{\partial u_1}{\partial n} \leq 0 & \text{on } \Gamma_C \quad \text{(exercise)} \\ \frac{\partial u_1}{\partial n} \cdot (u_1 - \varphi) = 0 \end{cases}$$

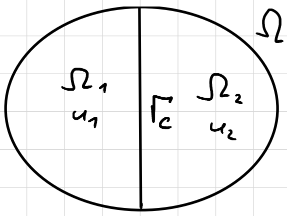


As long as $u_1 > \varphi$ (Material 1 does not touch the obstacle described by φ)
Material 1 is free to move as it likes ($\frac{\partial u_1}{\partial n} = 0$).
Once $u_1 = \varphi$, $\frac{\partial u_1}{\partial n} \leq 0$ (pushing downwards)

$L^\infty(\Gamma_c) \ni g \geq 0$

d) Friction: Energy $E(u_1,u_2) = \frac{1}{2}\int_{\Omega_1} (\nabla u_1)^2 + \frac{1}{2}\int_{\Omega_2} (\nabla u_2)^2 - \int_{\Omega_1} f_1 u_1 - \int_{\Omega_2} f_2 u_2 + \int_{\Gamma_c} g|u_1-u_2|$

car crash || wall



$\Updownarrow$

$\rightarrow \min$
$u_1,u_2 \in H^1(\Omega_{1/2})$
$u_1 = u_2$ on Γ_c
$u_1 = 0$ on $\partial\Omega$
$u_2 = 0$ on $\partial\Omega$

To see this (exercise):
$E(u_1+\lambda h_1, u_2+\lambda h_2) - E(u_1,u_2) \geq 0$
$\forall \lambda \forall h$ satisfying the conditions of "min"

$-\Delta u_1 = f_1$ in Ω_1

$-\Delta u_2 = f_2$ in Ω_2

$u_1 = u_2 = 0$ on $\partial\Omega$

Friction $\begin{cases} \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = 0 & \text{on } \Gamma_c \\ \left|\frac{\partial u_1}{\partial n}\right| \leq g & \end{cases}$ ← normal stress continuous

(if it's big, you start sliding)

$\frac{\partial u_1}{\partial n}(u_1-u_2) + g|u_1-u_2| = 0$ on Γ_c

II Well posedness + a basic Error Analysis of num. Discretizations

in most examples so far: $E(u) = \frac{1}{2}a(u,u) - \langle f,u\rangle$ where $u \in H$ Hilbert space e.g. $(H^1, H^1_0, \dots)$
(modulo friction)

$f \in H^*$ continuous lin. functional

$a(\cdot,\cdot)$ — " — , coercive, symmetric bilinear form

$|a(u,v)| \leq c\|u\|_H \|v\|_H$

$a(u,u) \geq \alpha \|u\|_H^2$

$a(u,v) = a(v,u)$

Theorem (Lax-Milgram):
$K \subseteq H$ closed & convex, $a(\cdot,\cdot)$ & f as above
$\Rightarrow E(u) \rightarrow \min$ has a unique solution

Proof: (1) $E(\cdot)$ is bounded from below

$E(v) = \frac{1}{2} a(v,v) - \langle f,v\rangle \geq \frac{\alpha}{2}\|v\|_H - \|f\|_{H^*}\|v\|$

(coercivity + continuity of f)

$= \frac{1}{2\alpha}\underbrace{(\alpha\|v\|_H - \|f\|_{H^*})^2}_{\geq 0} - \frac{\|f\|_{H^*}}{2\alpha} \geq -\frac{\|f\|_{H^*}}{2\alpha}$

E bounded $\Rightarrow \exists \inf$

let $c_1 = \inf_{v\in K} E(v) \geq -\frac{\|f\|^2_{H^*}}{2\alpha}$

let $v_n \in K$ be an infimizing sequence: $E(v_n) \rightarrow c_1$

② To show: v_n is a cauchy-sequence in H $\Rightarrow$ convergence

zz.: $\|v_n - v_m\|_H \to 0 \qquad n,m \to \infty$

$$\alpha \|v_n - v_m\|_H^2 \overset{\text{coercivity}}{\leq} a(v_n - v_m, v_n - v_m)$$

$$= 2a(v_n, v_n) + 2a(v_m, v_m) - a(v_n + v_m, v_n + v_m)$$

$$= 4E(v_n) + 4E(v_m) - 8E\Big(\underbrace{(v_n + v_m)\tfrac{1}{2}}_{\in K \text{ because } K \text{ is convex}}\Big)$$

$$\underbrace{\leq \underbrace{4E(v_n)}_{\to 4c_1} + \underbrace{4E(v_m)}_{\to 4c_1} - 8c_1}_{\xrightarrow{n,m\to\infty} 0}$$

③ Since K closed $\Rightarrow \lim_{n\to\infty} v_n =: u \in K$

E is continuous, because $a(\cdot,\cdot)$ and f continuous

$\Rightarrow E(u) = E(\lim_{n\to\infty} v_n) = \lim_{n\to\infty} E(v_n) = \inf E$

$\Rightarrow u$ is minimizer with $u \in K$

④ Uniqueness:

assume 2 minim. u_0, u_1. take $v_j = u_{j \bmod 2} = u_1, u_0, u_1, u_0, \dots$

from ② $\Rightarrow v_j$ is cauchy-sequ. $\Rightarrow \|v_n - v_{n+1}\| \to 0$

$\Rightarrow \|u_0 - u_1\| = 0 \Rightarrow u_0 = u_1$

■

Theorem (Characterization of Minimizer):

V vector space, a symmetric bilinear form on V
$a(v,v) > 0 \quad \forall v \neq 0$
$f: V \to \mathbb{R}$ linear $\quad C \subseteq V$ convex

Then u minimizes $E(v) = \frac{1}{2} a(v,v) - \langle f, v\rangle$ over C $\Leftrightarrow$ $(*)$ $a(u, v-u) \geq \langle f, v-u\rangle \; \forall v \in C$
(Variational Inequality)

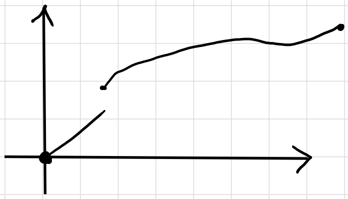
Remark: if C is a subspace of V: set $v = u + w \quad w \in C$

$$\left.\begin{aligned} &\Rightarrow (*) + a(u,w) \geq \langle f, w\rangle \\ &-a(u,-w) \geq \langle f, -w\rangle \Leftrightarrow a(u,w) \leq \langle f, w\rangle \end{aligned}\right\} \Rightarrow \boxed{a(u,w) = \langle f, w\rangle}$$

Proof: „$\Rightarrow$": $E(u + \lambda h) - E(u) = \lambda\big[a(u,h) - \langle f, h\rangle\big] + \frac{\lambda^2}{2} a(h,h) \qquad (**)$

set $h = v - u$ with $v \in C$

$\Rightarrow u + \lambda h = u + \lambda(v-u) \in C$ for $\lambda \in [0,1]$ because C is convex we know $(**) \geq 0$ because u minimizer



scope must be positive

$$\Rightarrow a(u,h) - \langle f,h \rangle \geq 0$$

$$\overset{\text{def. } h}{\Longleftrightarrow} \quad a(u,v-u) \geq \langle f, v-u \rangle \qquad (\text{this is } *)$$

„$\Leftarrow$": exercise

$\blacksquare$

Remark: The characterization Thm. + Lax-Milgram assure unique solutions to the transmission problem a), obstacle problem b), and the Signiori problem c). They characterize the solutions in terms of the variational inequality $(*)$ and $(*) \simeq$ weak formulation of the PDE.

A theorem of Lious-Stampacchia applies in the non-symmetric case.

<u>Theorem (Falk, Math. Comp., 1974):</u> $\approx$ Cea's lemma

- $a(\cdot,\cdot)$ symmetric, coercive ($\alpha \|u\|_H^2 \leq a(u,u)$), continuous ($|a(u,v)| \leq C \|u\|_H \|v\|_H$) bilinear form on H, $f \in H^*$
- $H_h \subseteq H$ finite dim. subspace
- $K_h \subseteq H_h$ convex, $K \subseteq H$ convex and closed

Let u_h be the minimizer of $E(v) = \frac{1}{2} a(v,v) - \langle f, v \rangle$ over K_h ($\exists! \, u_h \in K_h$)
u ——— " ——— " ——— " ——— " — K ($\exists! \, u \in K$ (generalization of Lax-Milgram))

Then for any Hilbert space W that satisfies $H \hookrightarrow W \underset{\text{cont}}{\overset{c}{\hookrightarrow}} H^*$ dense s.t. $(i(v),w)_{W^*} = (v,w)_H$

$$\|u-u_h\|_H^2 \leq \frac{C^2}{\alpha^2} \|u-v\|_H^2 + \frac{2}{\alpha} \|f - Au\|_W \cdot \left[\|u - v_h\|_{W^*} + \|u_h - v\|_{W^*} \right] \qquad \forall v \in K, \forall v_h \in K_h$$

provided $Au - f \in W$

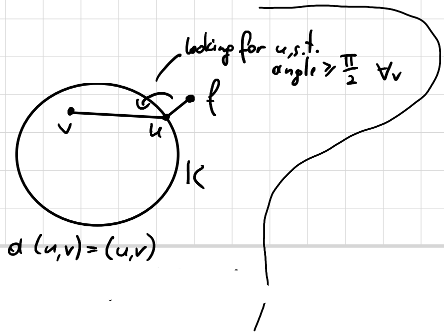
Some notation: $a(u,v) =: (Au, v)$, $A: H \to H^*$

$$\int |\nabla u|^2 \to -\Delta = A$$

Example: $a(u,v) = \frac{1}{2} \int_\Omega |\nabla u|^2$

$H = H_0^1(\Omega)$
$W = L^2(\Omega)$

<u>Proof:</u> Recall Characterization Thm.: u is the unique solution of: $a(u,u-v) \leq \langle f, u-v \rangle \quad \forall v \in K$
u_h — " ——— " —: $a(u_h, u_h - v_h) \leq \langle f, u_h - v_h \rangle \quad \forall v_h \in K_h$ $\quad \Big| +$



$$\Rightarrow a(u,u) + a(u_h,u_h) \leq \langle f, u-v \rangle + \langle f, u_h - v_h \rangle + a(u,v) + a(u_h, v_h) \qquad \Big| -2a(u,u_h)$$

$$\Rightarrow \alpha\|u-u_h\|_H^2 \underset{\text{coercivity}}{\le} a(u-u_h,u-u_h) \le \langle f,u-v_h\rangle + \langle f,u_h-v\rangle - a(u,u_h-v) - a(u,u-v_h) + a(u-u_h,u-v_h)$$

$$= (f-Au,u_h-v) + (f-Au,u-v_h) + a(u-u_h,u-v_h)$$

$$\le \|f-Au\|_W\left[\|u_h-v\|_{W*} + \|u-v_h\|_{W*}\right] + C\|u-u_h\|_H\|u-v_h\|_H$$

Young: $C\|u-u_h\|\|u-v_h\| \le \frac{\alpha}{2}\|u-u_h\|^2 + \frac{C^2}{2\alpha}\|u-v_h\|^2$

$$\Rightarrow \|u-u_h\|_H^2 \le \frac{2}{\alpha}\|f-Au\|_W\left[\dots\right] + \frac{C^2}{\alpha^2}\|u-v_h\|^2$$

∎

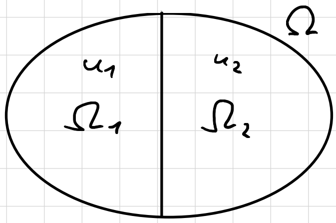
Remarks: See Falk for an application to the obstacle problem

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx \quad, u \ge \varphi$$

$H = H_0^1(\Omega)$

$W = L^2(\Omega)$ $\longrightarrow$ $\|u-u_h\| \le Ch^s$ for some $s>0$
for standard pw. polynomial H_h

<u>Long remark:</u> There are interesting interface problems that do not correspond to coercive bilinear forms a.



$$E(u_1,u_2) \overset{\text{Ex. }\alpha}{=} \frac{1}{2}\int_{\Omega_1}|\nabla u_1|^2 + \frac{\alpha}{2}\int_{\Omega_2}|\nabla u_2|^2 - \int_{\Omega_1} f_1u_1 - \int_{\Omega_2} f_2u_2$$

$$\begin{aligned} -\Delta u_1 &= f_1 && \text{in } \Omega_1\\ -\alpha\Delta u_2 &= f_2 && \text{in } \Omega_2\\ u_1-u_2 &= 0 && \text{on } \Gamma_c\\ \frac{\partial u_1}{\partial n} - \alpha\frac{\partial u_2}{\partial n} &= 0 && \text{on } \Gamma_c \end{aligned}$$

some bc on $\partial\Omega$ (e.g. Dirichlet)

a is coercive provided (the bc are good enough) and $\alpha > 0$.

For interesting novel metamaterials $\alpha < 0$.

<u>Assertion:</u> This problem is "elliptic" in a more general sense (in particular: The space of solutions is at most fin. dim.)
$\Leftrightarrow \alpha \neq -1$.

For $\alpha = -1$, hell breaks loose:

The space of solutions is infinite dim, or no solutions might exist.

To explain this:

Instead of transmission problems, we can focus on a system of PDEs in a fixed domain Ω

$$(*)\quad \begin{cases} Pu = f & \text{in } \Omega \quad \text{where } Pu = \begin{pmatrix} \vdots & \vdots \end{pmatrix} \quad p\times p \text{ matrix } = \sum_{|\alpha|\le 2} \underset{\substack{|\\ p\times p \text{ matrices}}}{a_\alpha(x)}\, \partial_x^\alpha u \\ Ru = g & \text{on } \partial\Omega \quad Ru = R_1 u|_\Gamma + R_2 \frac{\partial u}{\partial n}\Big|_\Gamma \end{cases}$$

where R_1 — ① C^∞ function
— ② Diff. op. of order ≤ 1 tangent to $\partial\Omega$

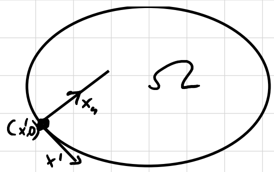
$R_2 = C^\infty$ function.

Interesting cases: $R_1 = 1$, $R_2 = 0$

Dirichlet problem

$R_2 \neq 0$: Neumann or Robin problem

Ellipticity: (Shapiro-Lopatinski) The boundary problem $(*)$ is called *elliptic* provided that the following initial-value problem has a unique solution $= 0$ $\forall\, |\xi'| = 1$ $\forall x \in \Gamma$

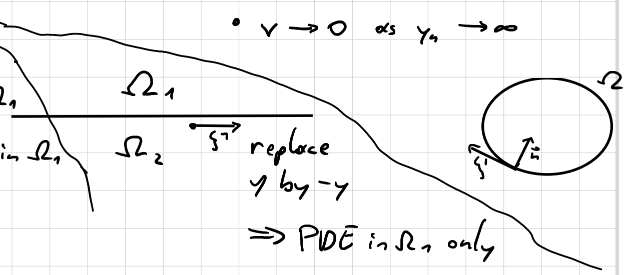


- $\displaystyle\sum_{|\alpha|\le 2} \overbrace{a_\alpha(x', x_n = 0)}^{= a_\alpha|_{x\in\Gamma}} \Big(\xi', \frac{\partial}{\partial y_n}\Big)^\alpha v(y_n)$

replace tangential derivatives by ξ'
keep normal derivative

- $\tilde{R}_1 v|_{y_n=0} + R_2 \frac{\partial v}{\partial y_n}\Big|_{y_n=0}$ $\quad y_n \geq 0$
- $v \to 0$ as $y_n \to \infty$

Example: Transmission problem:
(TP)



$-\Delta u_1 = f_1$ in Ω_1

$-\alpha\Delta u_2 = f_2(x,-y)$ in Ω_1

replace y by $-y$
$\Rightarrow$ PDE in Ω_1 only

$u_1 - u_2 = 0$

$\frac{\partial u_1}{\partial n} - \alpha \frac{\partial u_2}{\partial n} = 0$ on x-axis

$$P = \begin{pmatrix} -\Delta & 0 \\ 0 & -\alpha\Delta \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & -1 \\ \frac{\partial}{\partial n} & \alpha\frac{\partial}{\partial n} \end{pmatrix}$$

To check that TP is elliptic: $|\xi| = 1$

ODE: $(-\partial_y^2 - \xi^2)v_1(y) = 0 \longrightarrow v_1(y) = \cancel{A_1 e^{\xi y}} + B_1 e^{-\xi y}$ $\quad \not\to 0$

$(-\alpha\partial_y^2 - \alpha\xi^2)v_2(y) = 0 \qquad v_2(y) = \cancel{A_2 e^{\xi y}} + B_2 e^{-\xi y}$

$$v_1(0) - v_2(0) = 0, \quad \frac{\partial v_1}{\partial y}(0) + \alpha\frac{\partial v_2}{\partial y}(0) = 0$$

$A_1 = A_2 = 0$ because v_1 and v_2 should $\to 0$ as $y \to \infty$

$v_1(0) - v_2(0) = B_1 - B_2 = 0 \Rightarrow B_1 = B_2$

$-\xi B_1 - \xi\alpha B_2 = 0$

$-\xi B_1(1+\alpha) \overset{!}{=} 0$ $\quad\Rightarrow\quad B_1 = B_2 = 0$ unless $\alpha = -1$

<u>Conclusion:</u> (TP) is elliptic in the sense of Shapiro-Lopatinski

$\iff \alpha \neq -1$

<u>Theorem (well posedness):</u>

Assume $(*)$ is elliptic in a bounded smooth domain Ω.

Then the space of solutions to $(*)$ is at most finite-dim. It is nonempty for f, g outside finite-dimensional subspace.

Also: $\|u\|_{H^s(\Omega)} \leq C \|f\|_{H^{s-2}} + \|g\|_{H^{s-\mu}} + \|u\|_{H^{s-1}}$ for $s > \mu$, where $\mu = \frac{1}{2}$ (Dirichlet), $\frac{3}{2}$ in general