Rewrite as nonlinear equation. Solve with (semismooth) Newton method.

Abstract problem: $\begin{cases} \mathbb{F}(u,\lambda) = 0 \\ \lambda \geq 0, \quad u \geq 0, \quad \lambda \cdot u = 0 \end{cases}$ $\qquad \lambda = -\lambda_h$, $u = (u_h - \varphi)$ $\qquad \mathbb{F}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$

$$\Leftrightarrow \begin{cases} \mathbb{F}(u,\lambda) = 0 \\ \Phi(u,\lambda) = 0 \end{cases}$$

Exercise: $\lambda \geq 0, u \geq 0, u \cdot \lambda = 0 \Leftrightarrow \Phi(u,\lambda) = 0$ where

1.) $\Phi_0(u,\lambda) = u + \lambda - \sqrt{u^2 + \lambda^2}$ Fischer – Burmeister nonlinear complementarity function

2.) $\Phi_1(u,\lambda) = \max\{0, u\} \cdot \max\{u, 0\}$

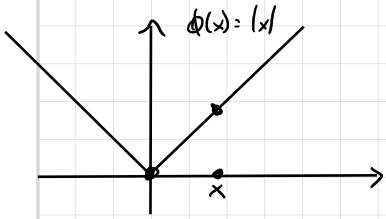
3.) $\Phi_\mu(u,\lambda) = (1-\mu)\Phi_0 + \mu \Phi_1 \qquad \mu \in (0,1)$

$\Phi_0, \Phi_1, \Phi_\mu \notin C^1$, but Lip.

Let $\Phi: \mathbb{R}^N \to \mathbb{R}^m$

Theorem: (Rademacher) If Φ Lipschitz $\Rightarrow$ For almost every x: Φ is differentiable at

Definition: The Clark subdifferential of Φ at x is the set $\partial\Phi(x) = \text{conv}\{H : \exists$ sequence (x_k) of points, where Φ' exists s.t. $x_k \to x$ and $\Phi'(x_k) \to H\}$



$\partial\Phi(x) = \{1\} \quad , x > 0$
$\partial\Phi(x) = \{-1\} \quad , x < 0$

$\partial\Phi(0) = \text{conv}\{+1, -1\}$
$\quad\quad = [-1, 1]$

Mi 8.6.2016

Newton method for non-smooth functions

Yesterday: Variational inequality

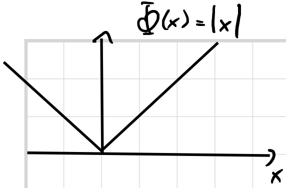
$\Updownarrow$

nonlinear, non smooth equation

$\mathbb{F}(u,\lambda) = 0 \leftarrow$ FE equation
$\varphi(u,\lambda) = 0 \leftarrow$ inequality constraints

φ involved e.g. $\max\{\cdot,\cdot\}$ or $|\cdot|$: not differentiable, only Lipschitz.

Aim: Do Newton's method on $\begin{cases} \mathbb{F} = 0 \\ \varphi = 0 \end{cases}$!



$$\Phi'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Clark subdifferential gives meaning to "$\Phi'(0)$":

$$\partial\Phi(x) = \text{conv}\{ H : \exists \text{ sequence } (x_k) \text{ of points where } \Phi'(x_k) \text{ exists}, x_k \to x \text{ and } \lim_{x_k \to x} \Phi'(x_k) = H \}$$

$$\partial\Phi(x) \overset{\text{example above}}{=} \begin{cases} \{\Phi'(x)\}, & x \neq 0 \\ [-1,+1], & x = 0 \end{cases}$$

Newton's method will be defined for the following class of fcts:

Def: Let $\Phi : \mathbb{R}^N \to \mathbb{R}^m$ Lipschitz in a neighbourhood of x.
Φ is called semi-smooth at x if all one-sided directional derivatives of Φ exist at x and $\forall d_k \to 0$ $\forall H_k \in \partial\Phi(x+d_k)$

$$\|\Phi(x+d_k) - \Phi(x) - H_k d_k\| \to 0 \quad \text{as} \quad k \to \infty$$

Examples: $\Phi_0(x) = \max\{0, x\}$

$\Phi_1(x) = |x|$

$\Phi_\mu(x) = (1-\mu)\Phi_0(x) + \mu\,\Phi_1(x)$

Newton's method to solve $\Phi(x) = 0$

- Choose initial guess x^0

For $k = 0, 1, 2, 3, \ldots$

- Choose $H_k \in \partial\Phi(x_k)$
- Solve $H_k s^k = -\Phi(x_k)$ (if H_k invertible)
- Set $x^{k+1} = x^k + s^k$

end

Theorem: Assume $\forall \tilde{x}$ in a neighbourhood of the solution x

$\|H^{-1}\| < C \quad \forall H \in \partial\Phi(\tilde{x})$ and that Φ is semi-smooth at x

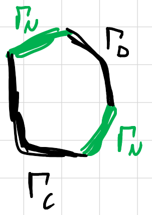
Then Newton's method converges superlinearly to x, provided $\|x^0 - x\|$ is small enough.

$$\frac{\|x^{k+1} - x\|}{\|x^k - x\|} \to 0$$
$\Leftrightarrow$ $x^k \to x$ superlinearly

In practice, one might use damped Newton's method, e.g., $x^{k+1} = x^k + \varsigma s^k$, $0 < \varsigma < 1$ and use numerical approximations to $\partial\Phi(x_k)$ (inexact Newton's method)

9.6.2016

Today:
- Contact problems for non convex materials / microstructure / phase transitions
- hp-adaptivity
- computable error estimates for a Signorini problem



$\overline{\Gamma_N} \cup \overline{\Gamma_c} =: \overline{\Gamma_\Sigma}$

Mixed problem for Signorini contact in Ω:

(MP)
$$a(u,v) - \langle \lambda, v\rangle_{\Gamma_c} = \langle f, v\rangle_\Omega \quad \forall v \in H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = u\}$$
$$\langle \mu - \lambda, u - g\rangle_{\Gamma_c} \geq 0 \quad \forall \mu \in \Lambda$$

Here $\Lambda := \{\lambda \in H^{-1/2}(\Gamma_c) : \int_{\Gamma_c} \lambda v \geq 0 \ \forall v \in H^{1/2}(\Gamma_c), v \leq 0\}$

i.e. "$\lambda \leq 0$"

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v$$

(MP_h)
$$a(u_h, v_h) - \langle \lambda_h, v_h\rangle_{\Gamma_c} = \langle f, v_h\rangle_\Omega \quad \forall v_h \in H_h \subseteq H^1_{\Gamma_D}(\Omega)$$
$$\langle \mu_h - \lambda_h, u_h - g\rangle_{\Gamma_c} \geq 0 \quad \forall \mu_h \in \Lambda_h.$$

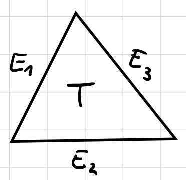
For the discretization, we fix a regular triangular mesh: $\mathcal{T}_h = \{T\}$, $\bigcup_{T \in \mathcal{T}_h} \overline{T} = \overline{\Omega}$

regular: in particular: h_T = height of T

C_1 (length of any side) $\leq h_T \leq C_2$ (length of any side)

no shallow or very obtuse angles

E_j edges of T



Assume $\overline{\Gamma_D} = \bigcup_{E \subseteq \Gamma_D} E$, $\overline{\Gamma_c} = \bigcup_{E \subseteq \Gamma_c} E$, $\overline{\Gamma_N} = \bigcup_{E \subseteq \Gamma_N} E$

$$H_h := \{v_h \in C^0(\overline{\Omega}) : \forall T \in \mathcal{T}_h : v_h|_T \text{ linear polynomial}, v_h|_{\Gamma_D} = 0\} \subseteq H^1_{\Gamma_D}(\Omega)$$
$$\Lambda_h := \{\mu_h : \mu_h = v_h|_{\Gamma_c} \text{ for some } v_h \in H_h, \int_{\Gamma_c} \mu_h w_h|_{\Gamma_c} \geq 0 \ \forall w_h \in H_h, w_h \geq 0\}$$

Proposition: Discrete problem reduces to

I mass matrix

$(I\vec{\lambda})_i \leq 0 \qquad \forall$ nodes $i \in \Gamma_c$

$(\vec{u} - g(x_i))_i \leq 0$

$(I\lambda)_i (\vec{u} - g(x_i))_i = 0$

Note that: • by choosing $\mu = 0$: $-\int_{\Gamma_c} \lambda(u-g) \geq 0$

$\mu = 2\lambda$: $+\int_{\Gamma_c} \lambda(u-g) \geq 0$

$$\Rightarrow \boxed{\int_{\Gamma_c} \lambda(u-g) = 0 \quad , \text{ similarly } \int_{\Gamma_c} \lambda_h(u_h - g) = 0}$$

• as we saw before: $\boxed{\lambda = \frac{\partial u}{\partial n} \\ u \leq g}$ on Γ_c by the equivalence of (MP) to the original PDE.

Aim: Derive a computable estimate: $\|u-u_h\|_{H^1(\Omega)} \leq C(\cdots \text{computable} \cdots)$

Coercivity: $\alpha \|u-u_h\|^2_{H^1(\Omega)} \leq a(u-u_h, u-u_h)$

$$= a(u-u_h, u-v_h) + a(u-u_h, v_h-u_h) \qquad \forall v_h \in H_h$$

$$= a(u, u-v_h) - a(u_h, u-v_h) + a(u-u_h, v_h-u_h)$$

where $a(u, u-v_h) \overset{(MP)}{=} \langle f, u-v_h\rangle_\Omega + \langle \lambda, u-v_h\rangle_{\Gamma_c}$ and, by (MP)-(MP$_h$),

$a(u-u_h, v_h-u_h) = \langle f, v_h-u_h\rangle + \langle \lambda, v_h-u_h\rangle_{\Gamma_c} - \langle f, v_h-u_h\rangle_\Omega - \langle \lambda_h, v_h-u_h\rangle_{\Gamma_c} = \langle \lambda - \lambda_h, v_h - u_h\rangle$

$$= \langle f, u-v_h\rangle - a(u_h, u-v_h) + \langle \lambda, u-v_h\rangle + \langle \lambda-\lambda_h, v_h-u_h\rangle$$

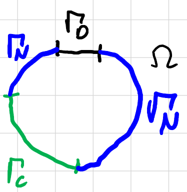
9.6.2016 NM

Aim: $\|u-u_h\|_{H^1(\Omega)} \leq$ computable.

see also: Hild, Llenas, Residual error estimates for Coulomb friction, SIAM J. Numer. Anal. 2009

(u,λ) solves mixed problem: $u \leq g$, $\lambda = \frac{\partial u}{\partial n} \leq 0$, $\int_{\Gamma_c} \lambda(u-g) = 0$

(u_h, λ_h) solves discretized mixed problem: $u_h \leq g$, $\lambda_h \leq 0$, $\int_{\Gamma_c} \lambda_h(u_h - g) = 0$



This morning: $\alpha \|u-u_h\|^2 \leq a(u-u_h, u-u_h)$

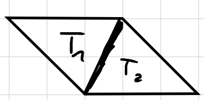
$$\overset{(MP),(MP_h)}{=} \cdots = \langle f, u-v_h\rangle_\Omega + \langle \lambda, u-v_h\rangle_{\Gamma_c} - a(u_h, u-v_h) - \langle \lambda_h - \lambda, v_h - u_h\rangle_{\Gamma_c}$$

$$= \langle f, u-v_h\rangle_\Omega - \langle \lambda_h, v_h - g\rangle - \langle \lambda, u_h - g\rangle - a(u_h, u-v_h) \quad \forall v_h \in H_h$$

$$a(u_h, u-v_h) = \int_\Omega (\nabla u_h)\cdot \nabla(u-v_h) \qquad \text{Green's formula} \quad \Delta u_h|_T = 0$$

$$= \sum_{T\in\mathcal{T}_h} \int_T (\nabla u_h)\cdot\nabla(u-v)$$

$$= -0 + \sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u_h}{\partial n}(u-v_h)$$



$$= \sum_{\substack{E \\ \text{interior } E}} \int_E \left(\frac{\partial u_h}{\partial n_E}\Big|_{\text{left of } E} - \frac{\partial u_h}{\partial n_E}\Big|_{\text{right of } E} \right)(u-v_h) + \sum_{E \subseteq \Gamma_N \cup \Gamma_c} \int_E \left(\frac{\partial u_h}{\partial n}\right)(u-v_h)$$

$$[\![\frac{\partial u_h}{\partial n}]\!]_E := (\;\cdots\;)$$

$$= \langle f, u-v_h\rangle_\Omega - \langle \lambda_h, v_h - g\rangle - \langle \lambda, u_h - g\rangle - \sum_{\text{interior } E} \int_E [\![\frac{\partial u_h}{\partial n_E}]\!](u-v_h)$$

$$- \sum_{E \subseteq \Gamma_N \cup \Gamma_C} [\![\frac{\partial u_h}{\partial n_E}]\!](u-v_h)$$

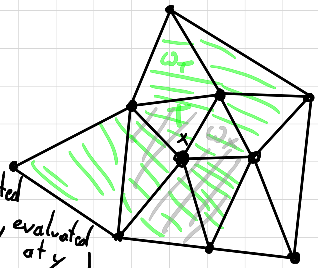
$$= \underbrace{\langle f, u-v_h\rangle}_{=:I} - \underbrace{\sum_{E \text{ interior or } E\subseteq\Gamma_N} [\![\frac{\partial u_h}{\partial n_E}]\!](u-v_h)}_{=:II} \underbrace{- \langle \lambda, u_h - g\rangle_{\Gamma_C} - \langle \lambda_h, u-g\rangle}_{=:IV} + \underbrace{\sum_{E\subseteq\Gamma_C}\int_E (\lambda_h - \frac{\partial u_h}{\partial n_E})(u-v_h)}_{=:III}$$

How to choose v_h? We use a (quasi) interpolation operator

$$\Pi_h : H^1_{\Gamma_D}(\Omega) \to H_h$$

"Clement-interpolation": $\Pi_h v(y) := \sum_{\text{Nodes } x \text{ not on } \Gamma_D} \left(\frac{1}{\text{Area}(\omega_x)} \int_{\omega_x} v(z)\,dz\right) \cdot$ (Hat fct. associated to node x, evaluated at y)

↖ number



Lemma: (Braess)

$$\|\Pi_h v - v\|_{L^2(T)} \le h_T \|\nabla v\|_{L^2(\omega_T)} \quad \forall v \in H^1_{\Gamma_D}$$

$$\|\Pi_h v - v\|_{L^2(E)} \le h_E^{1/2} \|\nabla v\|_{L^2(\omega_E)}$$

Proof: As in Braess. ∎

Choose: $v_h = u_h + \Pi_h(u-u_h)$

Estimate I to IV with this v_h

$$I = \int_\Omega f(u-v_h) = \sum_T \int_T f(\underbrace{u-u_h}_{v} - \Pi_h(\underbrace{u-u_h}_{v}))$$

$$\le \sum_T \|f\|_{L^2(T)} \|v - \Pi_h v\|_{L^2(T)} \overset{\text{Lemma}}{\lesssim} \sum_T h_T \|f\|_{L^2(T)} \|\nabla(u-u_h)\|_{L^2(\omega_T)}$$

($v = u-u_h$)

$$\le \frac{C}{2\delta} \cdot \sum_T h_T^2 \|f\|^2_{L^2(T)} + \frac{\delta \tilde{C}}{2} \|u-u_h\|^2_{H^1(\Omega)} \quad \forall \delta>0 \quad \text{(Young's inequality)}$$

$$\forall \delta>0 \quad I \le \frac{C}{2\delta} \sum_T h_T^2 \|f\|^2_{L^2(T)} + \frac{\delta\tilde{C}}{2}\|u-u_h\|^2_{H^1(\Omega)}$$

$$II = \sum_{E \text{ interior or } E\subseteq\Gamma_N} [\![\cdots]\!](u-v_h)$$

$$\le \sum \|[\![\cdots]\!]\|_{L^2(E)} \underbrace{\|u-v_h\|_{L^2(E)}}_{= \|u-u_h - \Pi_h(u-u_h)\|_{L^2(E)} \overset{\text{Lemma}}{\le} C h^{1/2} \|\nabla(u-u_h)\|_{L^2(\omega_E)}}$$

$$\le \frac{C}{2\delta} \sum_{E \text{ interior or } E\subseteq\Gamma_N} h_E \|[\![\frac{\partial u_h}{\partial n_E}]\!]\|^2_{L^2(E)} + \tilde{C}\delta \|u-u_h\|^2_{H^1(\Omega)}$$

$$III \le \frac{C}{2\delta} \sum_{E\subseteq\Gamma_C} h_E \|\lambda_h - \frac{\partial u_h}{\partial n_E}\|^2_{L^2(E)} + \tilde{C}\delta \|u-u_h\|^2_{H^1(\Omega)}$$

To sum up: $(\alpha - \delta(\dots))\, \|u-u_h\|^2_{H^1(\Omega)} \leq \tilde{C}(\delta)\Big(\sum_T h_T^2 \|f\|^2_{L^2(T)} + \sum_{\substack{E \text{ interior} \\ \text{or } E\subseteq\Gamma_N}} h_E \|[\![\frac{\partial u_h}{\partial n_E}]\!]\|^2_{L^2(E)} + \sum_{E\subseteq\Gamma_C} h_E \|\lambda - \frac{\partial u_h}{\partial n_E}\|^2_{L^2(E)}\Big)$ + IV

(computable)

$IV = -\langle \lambda, u_h - g\rangle - \langle \lambda_h, u-g\rangle$

As $\lambda \leq 0$ and $u_h - g \leq 0 \Rightarrow -\langle \lambda, u_h - g\rangle \leq 0$

As $\langle \lambda_h, u_h - g\rangle = 0 \Rightarrow -\langle \lambda_h, u-g\rangle = -\langle \lambda_h, u-u_h\rangle = \langle \lambda_h, u_h - u\rangle = \langle \lambda_{h+} + \lambda_{h-}, u_h - u\rangle = \langle \lambda_{h+}, u_h - u\rangle + \langle \lambda_{h-}, u_h - g\rangle - \langle \lambda_{h-}, u-g\rangle$

($-\langle \lambda_{h-}, u-g\rangle \leq 0$)

$\leq \langle \lambda_{h+}, u_h - u\rangle + \langle \lambda_{h-}, u_h - g\rangle$

$\leq \sum_{E\subseteq\Gamma_C}\Big(\|\lambda_{h+}\|_{L^2(E)} \|u_h - u\|_{L^2(E)} + \int_E \lambda_{h-}(u_h - g)\Big)$

Young's inequ.: $\leq \frac{c}{\delta}\sum_{E\subseteq\Gamma_C} \|\lambda_{h+}\|^2_{L^2(E)} + \sum_{E\subseteq\Gamma_C}\int_E \lambda_{h-}(u_h - g) + c\,\delta\, \|u_h - u\|^2_{H^1(\Omega)}$

(to the left hand side)

Theorem: $(\alpha - \delta(\dots))\|u_h - u\|^2_{H^1(\Omega)} \leq \tilde{\tilde{C}}(\delta)\Big(\dots\text{computable}\dots + \sum_{E\subseteq\Gamma_C} \|\lambda_{h+}\|^2_{L^2(E)} + \sum_{E\subseteq\Gamma_C}\int_E \lambda_{h-}(u_h - g)\Big)$ (*)

(computable)

The right hand side of (*) associates to every $T \in \mathcal{T}_h$ a number. If this number is big, you divide the triangle into 4 new ones. Solve the problem on the new mesh. Repeat until convergence.

Adaptive mesh refinements