3 General results concerning variational inequalities

Let V be a real Hilbert space with the inner product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|$. Let $V*$ denote the dual of V. Moreover, we assume

1. $a(\cdot,\cdot): V \times V \to \mathbb{R}$ is a continuous, V-elliptic bilinear form, i. e. $\exists \alpha > 0\ \forall v \in V:$ $a(v,v) \geq \alpha \|v\|^2$.
2. $L: V \to \mathbb{R}$ is a continuous linear functional.
3. $K \neq \emptyset$ is a closed convex subset of V.

Then the variational inequality reads
Find $u \in K$, such that

$$a(u, v-u) \geq L(v-u)\ \forall v \in K \tag{P1}$$

Theorem 3.1. *Lions,Stampacchia(Existence and Uniqueness)*
Problem (P1) possesses exactly one solution.

Proof. Uniqueness: Let $u_1, u_2 \in K$ be solutions of (P1):

$$\Rightarrow (a(u_1, v-u_1) \geq L(v-u_1)\ \forall v \in K \tag{3.1}$$

$$(a(u_2, v-u_2) \geq L(v-u_2)\ \forall v \in K \tag{3.2}$$

Set $v = u_2$ in (3.1) and $v = u_1$ in (3.2) and add them up. Applying (3.1)-(3.2) gives

$$\alpha\|u_1 - u_2\|^2 \leq a(u_1-u_2, u_1-u_2) \leq 0$$
$$\Rightarrow\ u_1 \equiv u_2$$

Existence: The Riesz representation theorem teaches us, that $\exists A \in \mathcal{L}(V,V),\ l \in V$ mit

$$\begin{aligned} (Au, v) &= a(u,v)\ \forall u,v \in V \\ L(v) &= (l,v)\ \forall v \in V \end{aligned} \tag{3.3}$$

(P1) is equivalent to: Find $u \in V$, such that

$$(u - \varrho(Au - l) - u, v-u) \leq 0\ \forall v \in K, \varrho > 0 \tag{3.4}$$

Remember: (3.4) is equivalent to:
Find $u \in V$: $u = P_K(u - \varrho(Au - l))$ for some $\varrho > 0$, where $P_K: V \to K$ is a projection with respect to $\|\cdot\|$.
Let us study the mapping $W_\varrho: V \to V$ defined via $W_\varrho(v) := P_K(v - \varrho(Av - l))$ and take $v_1, v_2 \in V$. It holds

$$\begin{aligned} \|W_\varrho(v_1) - W_\varrho(v_2)\|^2 &= \|P_K(v_1 - \varrho(Av_1 - l)) - P_K(v_2 - \varrho(Av_2 - l))\|^2 \\ &\leq \|(v_1 - v_2) - \varrho A(v_1 - v_2)\| \qquad (*) \\ &= \|v_2 - v_1\|^2 + \varrho^2\|A(v_2 - v_1)\|^2 \quad \underbrace{-2\varrho a(v_2 - v_1, v_2 - v_1)}_{-\varrho(A(v_1-v_2),(v_1-v_2))-(v_1-v_2,\varrho A(v_1-v_2))} \end{aligned}$$

(*) → see next page

and therefore

$$\Rightarrow \|W_\varrho(v_1) - W_\varrho(v_2)\|^2 \leq (1 - 2\varrho\alpha + \varrho^2\|A\|^2)\|v_1 - v_2\|^2$$
$$\Rightarrow W_\varrho \text{is a } \underline{\text{real}} \text{ contraction for } 0 < \varrho < \frac{2\alpha}{\|A\|^2}.$$

Thus there exists exactly one solution of the fixed point problem (Banach) $W_\varrho(v) = v$ and we have exactly one solution for problem (P1) □

For $(*)$ on the previous page, we use the Lemma below.

Let $K \subset V$ be a closed convex subset. For $x \in V$, denote by $P_K x$ the point closest to x in K:

$$\|x - P_K x\|_V = \min_{y \in K} \|x - y\|_V \ .$$

Note that according to the Lax-Milgram Theorem for closed convex subsets, there exists a unique such point $P_K x$. By the Characterization Theorem, it is determined by the variational inequality

$$(P_K x, y - P_K x)_V \geq (x, y - P_K x)_V \quad \forall y \in K \ . \tag{1}$$

Lemma 1. *P_K is a contraction: For $u, v \in V$: $\|P_K u - P_K v\|_V \leq \|u - v\|_V$.*

Proof. The function

$$d(t) = \|(1-t)P_K u + tu - ((1-t)P_K v + tv)\|_V^2 = \|P_K u - P_K v + t(u - v - P_K u + P_K v)\|_V^2$$

satisfies

$$d'(t) = 2(P_K u - P_K v, u - v - P_K u + P_K v)_V + 2t\|u - v - P_K u + P_K v\|_V^2 \ .$$

From (1) with $x = u$ and $y = P_K v$, $(P_K u - u, P_K v - P_K u)_V \geq 0$. Similarly, with $x = v$ and $y = P_K u$, $(P_K v - v, P_K u - P_K v)_V \geq 0$. Therefore $d'(t) \geq 0$ for $t \geq 0$, and $\|P_K u - P_K v\|_V^2 = d(0) \leq d(1) = \|u - v\|_V^2$. □

3.1 The approximation problem

Approximation of V and K.

Assumption 1. *Given a parameter $k \to 0$ and a family $\{V_h\}_h$ of closed subspaces of V. Moreover we denote by $\{K_h\}_h$ a family of closed convex non empty subspaces of V, such that $K_h \subset V_h \ \forall h$ (in general it is does* **not** *hold, that $K_h \subset K$), where*

1. *If $\{v_h\}_h$ is bounded in V, and $v_h \in K_h \ \forall h \Rightarrow \ v_h \to v \in K$* ($v_h \rightharpoonup v$ in V)
2. *$\exists \chi \subset V$ with $\overline{\chi} = K$ and $\exists r_h : \chi \to K_h$, such that $\lim_{h\to 0} r_h v = v$ strongly in $V \ \forall v \in \chi$*

Approximation of (P1):

$$\text{Find } u_h \in K_h : a(u_h, v_h - u_h) \geq L(v_h - u_h) \ \forall v_h \in K_h \tag{P1h}$$

Theorem 3.2. (P1h) *possesses exactly one solution.*

Proof. Apply the same arguments as in Theoerm 3.1, just change V_h by V and K_h by K. □

Theorem 3.3 (convergence, Glowinski). *Using the Assumptions 1 for K and $\{K_h\}_h$ there holds*

$$\lim \|u_h - u\| = 0 \text{ with } u_h \text{ solving (P1h) and } u \text{ solving (P1)}$$

Proof. (in 3 steps)

1. a priori error estimator for $\{u_h\}_h$
2. weak convergence of $\{u_h\}_h$
3. strong convergence

1. Estimation of u_h:

$$\text{Show: } \exists c_1, c_2 > 0 \text{ independent of } h : \|u_h\|^2 \leq c_1\|u_h\| + c_2 \ \forall h \tag{3.5}$$

$$\begin{aligned} u_h \text{ solves (P1h)} &\Rightarrow a(u_h, v_h - u_h) \geq L(v_h - u_h) \ \forall v_h \in K_h && (3.6)\\ &\Rightarrow a(u_h, u_h) \leq \underbrace{a(u_h, v_h)}_{(Au_h, v_h)} - L(v_h - u_h) \ \forall v_h \in K_h && (3.7)\\ &\Rightarrow \alpha\|u_h\|^2 \leq a(u_h, u_h) \leq \|A\|\|u_h\|\|v_h\| + \|L\|(\|v_h\| + \|u_h\|) \ \forall v_h \in K_h && (3.8)\end{aligned}$$

Choose some fixed $v_0 \in \chi$ and let $v_h = r_h v_0 \in K_h$. Then using Assumption 1(2.) for K_h, we see, that $r_h v_0 \to v_0$ and thus there exists $m \in \mathbb{R}$ independent of h, such that, $\|v_n\| \leq m$. Applying (3.8), we have

$$\|u_h\|^2 \leq \frac{1}{\alpha}\{(m\|A\| + \|L\|)\|u_h\| + \|L\|m\} = c_1\|u_h\| + c_2 \Rightarrow \|u_h\| \leq c \ \forall u_h$$

and thus we have shown uniform boundedness.

2. <u>weak convergence:</u> (3.5) implies, that $\{u_h\}$ is uniformly bounded in V. Therefore, there exists a subsequence $\{u_{h_i}\} \in K_{h_i}$ converging weakly to $u^* \in V$, i. e. $u_{h_i} \rightharpoonup u^* \in V$. Using Assumption 1(1.) for $\{K_h\}_h$, we have $u^* \in K$.

   Show: u^* is a solution of (P1).

   We know, that

$$a(u_{h_i}, v_{h_i} - u_{h_i}) \geq L(v_{h_i} - u_{h_i}) \ \forall v_{h_i} \in K_{h_i}. \tag{3.9}$$

   Let $v \in \chi$ and $v_{h_i} = r_{h_i} v$. Using (3.9), we have

$$a(\underbrace{u_{h_i}}_{u^*(h_i \to 0)}, u_{h_i}) \leq a(u_{h_i}, v_{h_i}) - L(v_{h_i} - u_{h_i}) \tag{3.10}$$

$$= a(\underbrace{r_{h_i} v}_{\to v}, u_{h_i}) - L(r_{h_i} v - u_{h_i})$$

$$\Rightarrow \liminf_{h_i \to 0} a(u_{h_i}, u_{h_i}) \leq a(u^*, v) - L(v - u^*) \ \forall v \in \chi \tag{3.11}$$

$$\Rightarrow 0 \leq a(u_{h_i} - u^*, u_{h_i} - u^*) \leq a(u_{h_i}, u_{h_i}) - a(u_{h_i}, u^*) - a(u^*, u_{h_i}) + a(u^*, u^*), \tag{3.12}$$

$$\text{i e. } a(u_{h_i}, u^*) + a(u^*, u_{h_i}) - a(u^*, u^*) \leq a(u_{h_i}, u_{h_i})$$

$$a(u^*, u^*) + a(u^*, u^*) - a(u^*, u^*) \leq \lim_{h_i \to 0} a(u_{h_i}, u_{h_i}),$$

$$\text{i.e. } a(u^*, u^*) \leq \lim_{h_i \to 0} a(u_{h_i}, u_{h_i}) \tag{3.13}$$

   Using (3.11) and (3.13), we get

$$\Rightarrow a(u^*, u^*) \leq \lim_{h_i \to 0} a(u_{h_i}, u_{h_i}) \leq a(u^*, v) - L(v - u^*) \ \forall v \in \chi$$

$$\Rightarrow a(u^*, v - u^*) \geq L(v - u^*) \ \forall v \in \chi \ \ u^* \in K \tag{3.14}$$

   As χ lies dense in K (i. e. $\overline{\chi} = K$) and $a(\cdot, \cdot)$, L are continuous, we get

$$a(u^*, v - u^*) \geq L(v - u^*) \ \forall v \in K (u^* \in K) \tag{3.15}$$

   i. e. $u^* = u$ is a unique solution. As u is an accumulation point of $\{u_h\}_h$ in the weak topology of V, the sequence $\{u_h\}_h$ converges weakly to u.

3. <u>strong convergence:</u> $a(\cdot, \cdot)$ is V-elliptic (coersive), thus

$$\begin{aligned} 0 \leq \alpha \|u_h - u\|^2 &\leq a(u_h - u, u_h - u) \\ &= a(u_h, u_h) - a(u_h, u) - a(u, u_h) + a(u, u), \end{aligned} \tag{3.16}$$

   where u_h solves (P1h) and u solves (P1).

$$r_n v \in K \ \forall v \in \chi \text{ and } u_h \text{ solves(P1h)}$$

   Then there holds with $r_h \in K_h \ \forall v \in \chi$

$$a(u_h, u_h) \leq a(u_h, r_h v) - L(r_h v - u_h) \ \forall v \in \chi \tag{3.17}$$

It holds $u_h \rightharpoonup u$ in V $(h \to 0)$ and $r_h v \to v$ in V $(h \to 0)$ using Assumption 1(2.). Hence inserting (3.17) in (3.16) and letting $h \to 0$ $(\forall v \in \chi)$ yields

$$0 \leq \alpha \lim_{h \to 0} \|u_h - u\|^2 \leq a(u, v-u) - L(v-u) \tag{3.18}$$

As $\overline{\chi} = K$ and $a(\cdot,\cdot)$, L are continuous, (3.18) holds for all $v \in K$

Setting $v = u$ in (3.18) gives us $\lim_{h \to 0} \|u_h - u\|^2 = 0$.

□