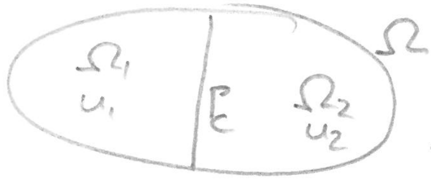
<u>Example d)</u> Consider $E(u_1,u_2) = \frac{1}{2}\int_{\Omega_1} (\nabla u_1)^2 + \frac{1}{2}\int_{\Omega_2} (\nabla u_2)^2$

$$- \int_{\Omega_1} f_1 u_1 - \int_{\Omega_2} f_2 u_2 + \int_{\Gamma_c} g\,|u_2 - u_1|$$

where



and $g \geq 0$.

Then if (u_1,u_2) minimizes $E(u_1,u_2)$ among all $u_1 \in H^1(\Omega_1)$, $u_2 \in H^1(\Omega_2)$, $u_1 = 0$ on $\partial\Omega$, $u_2 = 0$ on $\partial\Omega$, then

$$(*) \quad \begin{aligned} -\Delta u_1 &= f_1 \quad \text{in } \Omega_1 \\ -\Delta u_2 &= f_2 \quad \text{in } \Omega_2 \\ u_1 = u_2 &= 0 \quad \text{on } \partial\Omega \\ &\left.\begin{aligned} \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} &= 0 \\ |\vec{n}\cdot\nabla u_1| &\leq g \\ -\frac{\partial u_1}{\partial n}(u_2 - u_1) + g\,|u_2 - u_1| &= 0 \end{aligned}\right\} \text{on } \Gamma_c \end{aligned}$$

<u>Proof:</u> $E(u_1 + \lambda h_1, u_2 + \lambda h_2) - E(u_1,u_2)$

$$= \lambda\left\{\int_{\Omega_1} (\nabla u_1)(\nabla h_1) + \int_{\Omega_2} (\nabla u_2)(\nabla h_2) - \int_{\Omega_1} f_1 h_1 - \int_{\Omega_2} f_2 h_2\right\}$$

$$+ \int_{\Gamma_c} g\left(|u_2 - u_1 + \lambda(h_2 - h_1)| - |u_2 - u_1|\right) + \lambda^2(\cdots)$$

$$= -\lambda\left\{\int_{\Omega_1} (\Delta u_1 + f_1) h_1 + \int_{\Omega_2} (\Delta u_2 + f_2) h_2 - \int_{\Gamma_c} (\vec{n}\cdot\nabla u_1) h_1 + \int_{\Gamma_c} (\vec{n}\cdot\nabla u_2) h_2\right\}$$

$$+ \int_{\Gamma_c} g\left(|u_2 - u_1 + \lambda(h_2 - h_1)| - |u_2 - u_1|\right) + \lambda^2(\cdots)$$

$$\overset{!}{\geq} 0.$$

First note that for $h_1=0$ on Γ_C and $h_2=0$ on Γ_C:

$$\int_{\Omega_1} (\Delta u_1 + f_1)\, h_1 = 0 \qquad \forall\, h_1 \in H_0^1(\Omega_1)$$

and $$\int_{\Omega_2} (\Delta u_2 + f_2)\, h_2 = 0 \qquad \forall\, h_2 \in H_0^1(\Omega_2)$$

Therefore $-\Delta u_1 + f_1 = 0$ in Ω_1 and $-\Delta u_2 + f_2 = 0$ in Ω_2, as in $(*)$

Therefore $\forall\, h_{\substack{1\\2}} \in H^1(\Omega_{\substack{1\\2}})$, $h_{\substack{1\\2}} = 0$ on $\partial\Omega$, we obtain $\forall \lambda$

$$(**)\quad 0 \leq \int_{\Gamma_C} \Big(\lambda(\vec{n}\cdot\nabla u_1)h_1 - \lambda(\vec{n}\cdot\nabla u_2)h_2 + g\big(|u_2-u_1+\lambda(h_2-h_1)| - |u_2-u_1|\big)\Big)$$

First, assume that $h_1 = h_2$ on Γ_C. Then the g-term vanishes and

$$0 \leq \lambda \int_{\Gamma_C} (\vec{n}\cdot\nabla u_1 - \vec{n}\cdot\nabla u_2)\, h_1 \qquad \forall h_1\ \forall \lambda$$

$$\Longrightarrow \quad \vec{n}\cdot\nabla u_1 - \vec{n}\cdot\nabla u_2 = 0 \qquad \text{as claimed in } (*).$$

$(**)$ now becomes $\forall h_1\ \forall h_2$

$$0 \leq \int_{\Gamma_C} \big\{\lambda(\vec{n}\cdot\nabla u_1)(h_1-h_2) + g\big(|u_2-u_1+\lambda(h_2-h_1)| - |u_2-u_1|\big)\big\}$$

Setting $H := \lambda(h_2-h_1)$, this is

$$0 \leq \int_{\Gamma_C} \big\{-(\vec{n}\cdot\nabla u_1)H + g\big(|u_2-u_1+H| - |u_2-u_1|\big)\big\} \quad (***)$$

$$\leq \int_{\Gamma_C} \big\{-(\vec{n}\cdot\nabla u_1)H + g|H|\big\}$$

$$\overset{g\geq 0}{\Longrightarrow} \left|\int_{\Gamma_C} (n\cdot\nabla u_1)H\right| \leq \int_{\Gamma_C} g|H| \Longrightarrow |\vec{n}\cdot\nabla u_1| \leq g \quad \text{as in } (*)$$

The remaining assertion follows from $(***)$. $\boxtimes$