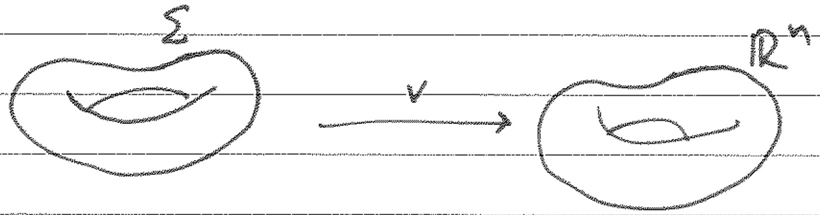
Variational Methods for PDEs Final Lecture

Nash: Def: (Σ, g) Riemannian Manifold

Σ: "n-dim surface" $\quad g$: metric



v short if the induced way of measuring lengths on Σ is $\leq g$

More precisely:

$$\underbrace{\left[v^{\#}\left(\underset{\text{on } \mathbb{R}^n}{\text{euclidean metric}}\right)\right]_{ij}}_{\substack{\text{scalar product for} \\ \text{tangent vectors on } \Sigma}} := (\partial_i \vec{v}) \cdot (\partial_j \vec{v})$$

If $\sum_{ij} g_{ij} \xi_i \xi_j \geq \sum_{ij} \left[v^{\#}(\text{eucl})\right]_{ij} \xi_i \xi_j \quad \forall \xi$

$\Longrightarrow$ v is short.

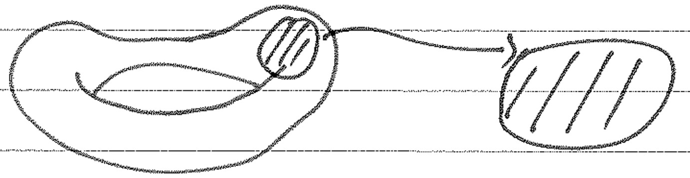
Thm: (Nash-Kuiper) (Σ, g) smooth, without boundary, of dim n

- $v: \Sigma \longrightarrow \mathbb{R}^N$, C^∞, short, $N \geq n+2$ s.t. Jacobian is injective (immersion)

$\Longrightarrow \forall \varepsilon > 0 \ \exists$ isometric C^1-immersion $u: \Sigma \longrightarrow \mathbb{R}^N$ s.t. $\|u - v\|_{C^0} < \varepsilon$.

Corollary: Any smooth n-dimensional Riemannian manifold has a C^1-isometric immersion in $\mathbb{R}^{2n}$.

While the statement & principal interest of these results are geometric, the underlying techniques are about nonlinear pdes.



$\subseteq \mathbb{R}^n$

g isometric to $v^{\#}(\text{eucl.}) = (\partial_i \vec{v}) \cdot (\partial_j \vec{v})$

$\iff$ Find $\vec{v}$ s.t. $\boxed{(\partial_i \vec{v}) \cdot (\partial_j \vec{v}) = \text{given}}$ $\quad \frac{n(n+1)}{2}$ equations for N unknowns

$\mathbb{R}^n \supseteq \Omega \to \mathbb{R}^N$

Claim: $\exists$ C^1 solution for $N = n+2$

References: C. de Lellis · Masterpieces of John F. Nash
· Nash's nonlinear iteration

The theorem will follow from:

Key proposition (NNI, 2.1)

· (Σ, g) as above

· $z : \Sigma \to \mathbb{R}^N$ C^∞ short immersion

$\Longrightarrow \forall \eta, \delta > 0$ $\exists$ C^∞ short $z_1 : \Sigma \to \mathbb{R}^N$ s.t.

· $\|z - z_1\|_{C^0} \le \eta$ $\leftarrow$ z_1 is close to z

eqn is almost satisfied $\rightarrow$ · $\|g - z_1^{\#}(\text{eucl.})\|_{C^0} \le \delta$

· $\|Dz_1 - Dz\|_{C^0} \le C \sqrt{\|g - z^{\#}(\text{eucl.})\|_{C^0}}$ $\quad \exists\, C = C(\Sigma)$

$Dz_1 - Dz$ is not too big

(If z injective $\Rightarrow$ z_1 injective)

This is what we will show.

Proof of Thm using key proposition:

Let ~~v~~ $v =: v_0$. May assume v_0 strictly short i.e. "$<$" instead of "$\leq$".

Apply proposition with $z = v_0$ and $\eta = \frac{\varepsilon}{4}$, $\delta = \frac{1}{4}$.

$\Rightarrow$ Get $z_1 =: v_1$ which is "more isometric" in the sense of the proposition.

Iterate $z = v_j$, $\eta = \frac{\varepsilon}{2^{j+2}}$, $\delta = \frac{1}{4^{j+1}} \longrightarrow z_1 = v_{j+1}$

i.e. apply proposition with smaller and smaller η's and δ's.

Take $\lim_{j\to\infty} v_j =: u$

From Δ-inequality: $u = \lim_{j\to\infty} v_0 + (v_1 - v_0) + (v_2 - v_1) + (v_3 - v_2) + \cdots$

$\|v_1 - v_0\| \leq \eta = \frac{\varepsilon}{2^2}$

$\|v_2 - v_1\| \leq \eta = \frac{\varepsilon}{2^3} \quad \Rightarrow \|v_{j+1} - v_j\|_{C^0} \leq \frac{\varepsilon}{2^{j+1}}$

$\Rightarrow$ u exists. (because the series cges)

$\|Dv_{j+1} - Dv_j\|_{C^0} \leq C(\Sigma)\sqrt{\|g - v_j^{\#}(\text{eucl.})\|_{C^0}} \leq C\delta^{1/2} \longrightarrow 0$

$\Rightarrow$ convergence is in C^1, not just in C^0

$(\partial_i \vec{u}) \cdot (\partial_j \vec{u}) \leftarrow v_k^{\#}(\text{eucl.}) = (\partial_i \vec{v}_k) \cdot (\partial_j \vec{v}_k) \xrightarrow[k\to\infty]{} g$

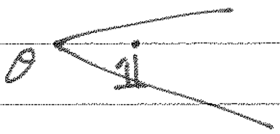
(because C^1-cgce)

$\Rightarrow \boxed{(\partial_i u) \cdot (\partial_j u) = g}$ have solved eqn. ∎

Local ("analysts'") statement of key proposition

Define $\mathcal{C}_r := \{A \in \mathbb{R}^{n\times n}_{sym} : \left\| \frac{A}{\frac{1}{n}\operatorname{tr} A} - \mathbb{1} \right\| < r\}$

cone of symmetric
$n\times n$ matrices near $\mathbb{1}$



Key prop: $\exists\, r_0 = r_0(n)$ s.t.: If $\mathcal{U} \subset \mathbb{R}^n$ bdd simply connected (no holes) open domain

- $g \in C^\infty(\bar{\mathcal{U}})$ smooth metric
- $z : \bar{\mathcal{U}} \to \mathbb{R}^N$ C^∞ smooth, short s.t. $g - z^\#(\text{eucl.}) \in \mathcal{C}_{r_0}$ $\forall x \in \bar{\mathcal{U}}$

$\Longrightarrow$ $\forall \eta, \delta > 0$ $\exists$ smooth short map $z_1 : \bar{\mathcal{U}} \to \mathbb{R}^N$ s.t.

$g - z_1 \in \mathcal{C}_{r_0}$ $\forall x \in \bar{\mathcal{U}}$

and $\|z - z_1\|_{C^0} \le \eta$

$\|g - z_1^*(\text{eucl.})\|_{C^0} \le \delta$

$\|Dz - Dz_1\|_{C^0} \le M_n \sqrt{\|g - z_1^*(\text{eucl.})\|_{C^0}}$

In order to prove this: (NNI, 2.4)

Lemma 2.4: Let $S_+^{n\times n}$ = set of pos. def. symmetric $n\times n$ matrices

$\Rightarrow$ $\exists$ neighborhood W of $\mathbb{1}$ and $N(n) = \frac{n(n+1)}{2}$ unit vectors $v_k \in \mathbb{R}^n$ s.t.

any $A \in W$ can be written as

$$A = \sum_{k=1}^{N} \lambda_k(A)\, v_k \otimes v_k$$

where $\lambda_k : W \to \mathbb{R}$ linear in A and $\lambda_k(A) \ge \rho_0(n) > 0$

Proof of analysts' key proposition ($\Rightarrow$ Theorem)

$$z^*(\text{eucl.}) = Dz^T Dz$$

By assumption $g - Dz^T Dz \in \mathcal{C}_{r_0}$ ← all of these matrices >0
$\Rightarrow g - Dz^T Dz \geq 2\gamma \mathbb{1} \quad \exists \gamma > 0.$
wlog $\gamma \leq \delta$, Let $h = g - Dz^T Dz - 2\gamma \mathbb{1} \in \mathcal{C}_{r_0}$

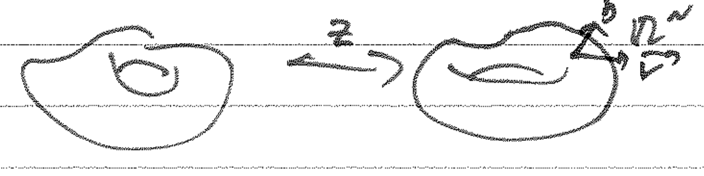
<u>From lemma</u>: $h(x) = \sum_{k=1}^{N} a_k^2(x)\, v_k \otimes v_k$ C^∞ because $\underset{d^2}{\overset{\lambda}{\shortparallel}}$ linear in h

Define improvement of z: $\bar{z}$

$$D\bar{z}^T D\bar{z} \simeq Dz^T Dz + h_1$$ This shrinks the error in $z^*(\text{eucl.}) = g$

The rigorous formulation is:
<u>Proposition</u>: Under the above assumptions
$\exists\, \bar{z} \in C^\infty(\bar{\Omega} \to \mathbb{R}^N)$ short s.t.
$\|\bar{z} - z\|_{C^0} < \tilde{\eta}$, $\|D\bar{z}^T Dz - a_1^2\, v_1 \otimes v_1\|_{C^0} \leq \tilde{\delta}$
$\|D\bar{z} - Dz\|_{C^0} \leq \tilde{M}_n \|a_1\|_{C^0}$

<u>Proof</u>: $\bar{z}(x) = z(x) + z_p(x)$ where z_p is defined as follows:



$N \geq n+2$
$\Rightarrow$ Can find 2 normal vectors which are $\perp$ $\forall x$.

$\|v\| = \|b\| = 1 \quad \forall x$

Define $z_p = \dfrac{a_1(x)}{\lambda}\left(v(x)\cos(\lambda v_1 \cdot x) + b(x)\sin(\lambda v_1 \cdot x)\right)$

λ chosen to make estimates work. $\gg 1$.
$\Rightarrow z - \bar{z}$ small for $\lambda \gg 1$.
$\|D\bar{z} - Dz\| \leq C\|a\|$ ✓

$$D\bar{z} = Dz + Dz_p$$

$$D\bar{z}^T D\bar{z} = (Dz + Dz_p)^T (Dz + Dz_p)$$

$$= Dz^T Dz + \underbrace{Dz^T Dz_p}_{=0} + \underbrace{Dz_p^T Dz}_{=0} + \underbrace{Dz_p^T Dz}_{\text{leading term}}$$

__Key observation__:

because corrugation z_p ~~were~~ were added in normal direction. $\blacksquare$