# Variational Methods for PDEs L12

$$\int_0^T \langle \hat{u}'_{\tau_n}(t), v\phi(t)\rangle_H \,dt + \int_0^T \langle DI(u^+_{\tau_n}), v\,\phi(t)\rangle\, dt = 0$$

where $\tau_n \to 0^+$. $\forall \phi \in C([0,T])$ $\forall v \in X$.

The product $\phi(t)v \in C([0,T], X) \subseteq L^p([0,T], X) \subseteq \left(L^{p'}([0,T], X^*)\right)^*$ $\forall p$

From a priori estimate:

$DI(u^+_{\tau_n})$ converges weakly in $L^{p'}([0,T], X^*)$

so $\int_0^T \langle DI(u^+_{\tau_n}), v\phi(t)\rangle\, dt$ converges.

to $\int_0^T \langle \xi, v\phi(t)\rangle\, dt$

Similarly, $\hat{u}'_{\tau_n} \in L^{p'}([0,T], X^*)$ and converges weakly

$$\Rightarrow \int_0^T \langle \hat{u}'_{\tau_n}(t), v\phi(t)\rangle\, dt \to \int_0^T \langle u'(t), v\phi(t)\rangle\, dt.$$

From discrete evolution equation,

$$0 = \int_0^T \langle \hat{u}'_{\tau_n}, v\phi(t)\rangle + \langle DI(\tau_n^+), v\phi(t)\rangle\, dt$$

$$\longrightarrow \int_0^T \langle u', v\phi(t)\rangle + \langle \xi, v\,\phi(t)\rangle\, dt \qquad \forall v, \phi$$

For $\phi \to$ Dirac sequence in time, we recover that $u$ satisfies the claimed version of the gradient flow eqn.

As $W^{1,p'}$ – functions are continuous (Sobolev embedding)

$$\hat{u}_{\tau_n}(0) \longrightarrow u(0)$$

Continuity of $u$ even implies $\lim_{t \to 0^+} u(t) = u_0$

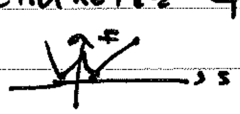
<u>Example:</u> $F \in C^1(\mathbb{R})$, $0 \leq F(s) \leq C_F(1+s^2)$

$|f(s)| = |F'(s)| \leq C'_F(1+|s|)$

$$I(u) = \frac{1}{2}\int_\Omega (\nabla u)^2 + F(u) \, ds$$

Dirichlet energy which leads to Laplace eqn.

For $F(s) = k^2 s^2 \Rightarrow$ Helmholtz equation

$F(s) = |s+1||s-1|$  (round it near corners)

We consider $X = H^1_0(\Omega)$, $H = L^2(\Omega)$, $p=2$.

$$\langle DI(u), h \rangle = \int_\Omega \nabla u \nabla h + \int_\Omega F'(u) h$$

Gradient flow: $\langle \frac{du}{dt}, v \rangle + \underbrace{\langle DI(u), v \rangle}_{\langle \nabla u, \nabla v \rangle + \langle F(u), v \rangle} = 0$

If $u$ is smooth enough, integrate by parts in the second term to conclude

$$\boxed{\frac{du}{dt} - \Delta u + F(u) = 0}$$

Theorem says that there exists a solution to

$$\langle u', v \rangle_* + \langle \xi, v \rangle = 0$$

where $u = \lim \hat{u}_{\tau_n}$, $\xi = DI(u_{\tau_n}) + \tau_n$

$$u^+ \longleftarrow \hat{u}^+_{\tau_n}, \quad \xi \longleftarrow D_i(u^+_{\tau_n})$$

Need to show that $\xi = DI(u)$

<u>A priori</u>: $u_{\tau_n}^{+} \rightharpoonup u$ in $L^2([0,T], \cancel{X}\, H_0^1(\Omega))$

$\{p\}$ $X = H_0^1(\Omega)$

$$\Rightarrow \nabla u_{\tau_n}^{+} \rightharpoonup \nabla u \text{ in } L^2([0,T], L^2(\Omega))$$

$$\Rightarrow \int_0^T \langle \nabla u_{\tau_n}^{+}, \nabla v \rangle \to \int_0^T \langle \nabla u, \nabla v \rangle \quad \forall v \in H_0^1(\Omega) = X$$

For the non linear part of $DI(u)$, we observe:

<u>Q</u>: $f(u_{\tau_n}^{+}) \cdots \xrightarrow{?} f(u)$

$\in L^2([0,T], X)$

$\hat{u}_{\tau_n} \in W^{1,2}([0,T], X^* = H_0^1(\Omega)^*)$

$\rightsquigarrow u_{\tau_n}^{+}$ point evaluation

$$\Rightarrow u_{\tau_n}^{+} \in L^2([0,T] \times \Omega)$$

Because embedding is compact, $u^{+}_{\tau_n} \to u(t)$ in $L^2([0,T], L^2(\Omega))$

$$\Rightarrow \int_0^T \int_\Omega \underbrace{f(u_{\tau_n}^{+})}_{\leq C_F'(1+|u^{+}_{\tau_n}|)} w \, dx \, dt$$

$$\xrightarrow{\text{LDCT}} \int_0^T \int_\Omega f(u) w \, dx \, dt$$

$$\Rightarrow \int_0^T \langle DI(u^{+}_{\tau_n}), w \rangle \xrightarrow{n \to \infty} \int_0^T \int_\Omega \nabla u \nabla w + \int_0^T \int_\Omega f(u) w$$

This allows us to refine our conclusion:

$\xi = DI(u)$, and therefore

$\exists$ an honest (weak) solution to the gradient flow $(*)$

$$\left\langle \frac{du}{dt}, v \right\rangle + \langle \nabla u, \nabla v \rangle + \langle f(u), v \rangle = 0$$

For convex $I$ one can refine the existence theory and improve on the a priori estimates, in particular getting rid of the requirement that $I$ is Frechet differentiable.

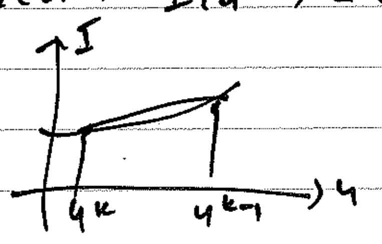
Example: $I(u) = \frac{1}{2}\int_\Omega (\nabla u)^2 + \int |u+g|$

"friction", non smooth minimization

<u>2 Key ideas:</u>

<u>Convexity</u>: $\forall\, u^k, u^{k-1}$:

$$I(u^k) - I(u^{k-1}) \leq \langle DI(u^k), u^k - u^{k-1} \rangle$$




flow eqn: $0 = \left\langle \frac{u^k - u^{k-1}}{\tau}, v \right\rangle + \langle DI(u^k), v \rangle$

$$v = \frac{u^k - u^{k-1}}{\tau} \Rightarrow = \left\langle \frac{u^k - u^{k-1}}{\tau}, \frac{u^k - u^{k-1}}{\tau} \right\rangle + \left\langle DI(u^k), \frac{u^k - u^{k-1}}{\tau} \right\rangle$$

$$\geq \frac{1}{\tau}\left(\tau\left\| \frac{u^k - u^{k-1}}{\tau}\right\|_H^2 + I(u^k) - I(u^{k-1})\right)$$

$$\Rightarrow I(u^k) + \underbrace{\left\| \frac{u^k - u^{k-1}}{\tau}\right\|_H^2}_{= \hat{u}'_\tau} \leq I(u^{k-1})$$

$\Rightarrow$ improved a priori estimate

i.e. $\boxed{\text{convexity} \Rightarrow \text{improved a priori estimate}}$

Also for convex functional $\partial I$ = subgradient

Thm 2.8: $\forall u_0 \in H$ s.t. $\partial I(u_0) \neq \emptyset$ and every $T > 0$
$\exists!$ solution to $\langle u', v\rangle + \langle \xi, v\rangle = 0$
with $\xi \in \partial I(u(t))$
If $I \in C^1 \Rightarrow \partial I = \{\text{singleton}\} \Rightarrow$ honest soln to grad. flow.

Final 1+2 lectures: Convex integration

1950s: Nash in Riemannian geometry
1970s (80s): Gromov in Topology "h-principle"
1990s: Vladimir Sverak & Stefan Müller: nonlinear elliptic PDEs
2009 - now: de Lellis, Szekelyhidi: fluid mechanics

Current interest: Incompressible Euler equations + Navier-Stokes

Convex integration constructs "wild" weak solutions to these equations

2 references:
- dL, S, "nontechnical"
- T. Tao, lecture notes on Onsager's conjecture

## Incompressible Euler equations:

$\Omega = \mathbb{R}^n \quad \mathbb{R}^n / \mathbb{Z}^n$ torus in $n$-dimensions (= periodic BC)

$u : [0,T] \times \Omega \to \mathbb{R}^n$ "velocity of the fluid at time $t$ in point $x$"

$p : [0,T] \times \Omega \to \mathbb{R}$

$$\left.\begin{aligned} \partial_t u + \overbrace{(u \cdot \nabla) u}^{\operatorname{div}(u \otimes u)} &= -\nabla_x p \\ \operatorname{div} u &= 0 \end{aligned}\right\} (*)$$

Observation: Kinetic energy $\frac{1}{2} \int_\Omega |u|^2 \, dx = E(t)$

$\partial_t E(t) = 0$ (formally differentiating under integral sign)

Onsager's conjecture: Let $u$ be a "weak solution" to $(*)$

Constantin, E, Titi (classical)

$\alpha \in (0,1)$, $n \geq 2$ and $0 < T < \infty$

(i) $\alpha > 1/3 \Rightarrow$ If $u \in C_t^0 C_x^{0,\alpha} \Rightarrow E(t)$ const any $\forall t \in [0,T]$

(ii) $\alpha \leq 1/3 \Rightarrow \exists \; u \in C_t^0 C_x^{0,\alpha}$ s.t. $E(t)$ is not constant.

Phil Isett (2018)

Such nonsmooth solutions relate to turbulent flows. The constructions will show further crazy properties e.g. nonunique solutions, solutions which are compactly supported in time.

Idea: ~~[illegible]~~ add corrugations