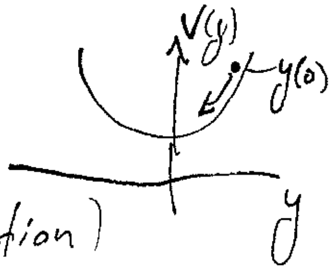
Variational Methods
26.02.2019

Gradient Flows: Idea: $V: \mathbb{R}^n \to \mathbb{R}$ $\quad C^2(\mathbb{R}^n)$

$$y' = -\nabla V(y), \; y(0) = y_0.$$



- Picard-Lindelöf: $\exists!$ solution.
- $\displaystyle\int_0^{T'} |y'|^2 dt + V(y(T')) = V(y(0))$ (Energy conservation)

$$\Rightarrow t \mapsto V(y(t)) \searrow$$

- Stationary points $\nabla V(y) = 0 \leftrightarrow$ longtime limits of grad. flows

Functional Analytic setup in ∞ dims:

- $L^p(0,T;X)$, $W^{1,p}(0,T;X)$ – Bochner spaces
- Evolution/Gelfand triple: H – Hilbert

$$X \xhookrightarrow[\text{dense}]{} H \hookrightarrow X^*$$

Fact: If $u \in L^p(0,T;X)$, $u' \in L^{p'}(0,T;X^*)$, then $u \in C([0,T];H)$.

$$\|u\|_{C(0,T;H)} \lesssim \|u\|_{L^p(0,T;X)} + \|u'\|_{L^{p'}(0,T;X^*)}$$

IBP: $\displaystyle\langle u(t_2), v(t_2)\rangle_H - \langle u(t_1), v(t_1)\rangle_H = \int_{t_1}^{t_2} {}_{X^*}\langle u'(t), v(t)\rangle_X + {}_{X^*}\langle v'(t), u(t)\rangle_X \, dt$

In particular: $\frac{1}{2}\partial_t \|u(t)\|_H^2 = {}_{X^*}\langle u'(t), u(t)\rangle_X$.

- Analogues of other embedding theorems also hold, e.g.:

$$L^p(0,T;X) \cap W^{1,q}(0,T;X^*) \xhookrightarrow[\text{cpt.}]{} L^p(0,T;H).$$

Let (X, H, X^*) be Gelfand triple. Do solutions to grad. flow exist?

$$\partial_t u = -\nabla_H I(u) \; ; \; u(0) = u_0 \qquad (*)$$

Here $I: X \to \mathbb{R}$ diffable in Fréchet sense.

$DI: X \to X^*$

$\nabla_H I(u) := DI(u)$

So, rigorously $(*)$ means:

$$(**) \quad {}_{X^*}\langle \partial_t u(t), v \rangle_X + {}_{X^*}\langle DI(u), v \rangle_X = 0 \qquad \forall v \in X$$

$u \in L^p(0,T; X)$ is a solution for given $u_0 \in H$ provided $\partial_t u \in L^{p'}(0,T; X^*)$ and $(**)$ satisfied for a.e. $t \in (0,T)$.

Ex: $I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$ on $X = H_0^1(\Omega)$ with $H = L^2(\Omega)$ leads to heat eq.

Def: I is called semicoercive (elliptic) if $\exists\, s > 0, c_1 > 0, c_2$ s.t.

$$I(v) \geq c_1 \|v\|_X^s - c_2 \|v\|_H^2$$

Ex: Helmholtz eq: $I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{k^2}{2}\int_\Omega |u|^2 dx.$

Not coercive!! Need c_2.

$$I(u) \geq c_1 \|u\|_{H^1}^2 - c_2 \|u\|_{L^2}^2.$$

<u>Proof of existence of solutions</u>: "Numerical" → • discretise in time
• solve elliptic problem
• let time step → 0.

<u>Hope</u>: The above idea converges to a solution to (**).

<u>Prop:</u> (Implicit Euler in time)

Let:
- $I : X \to \mathbb{R}$ semicoercive + wlsc.
- $0 < \tau < \frac{1}{4c_2}$
- $\forall k = 1, 2 \ldots, k = \lceil \frac{T}{\tau} \rceil$: $I^k(u) = \frac{1}{2\tau} \|u - u^{k-1}\|_H^2 + I(u)$

where $u^0 = u_0 \in H$.

$\Rightarrow \forall k \ \exists!$ minimizer u^k of I^k and $\left(\frac{u^k - u^{k-1}}{\tau}, v \right)_H + \langle DI(u^k), v \rangle_{X^* \times X} = 0$ $(**_\tau)$

i.e. $\exists$ discrete in time soln. to the grad. flow (**).

<u>Proof:</u> For small enough τ, $I^k(u)$ is coercive.
Thus unique minimizer exists. As I is diffable,
I^k is also diffable $\Rightarrow$ EL eq. is satisfied. ∎

General theory beyond minimisation problems ("non-variational")
Theory of monotone/pseudo-monotone operators

<u>Ref:</u> Zeidler, Nonlinear Functional Analysis

We now show some apriori bounds for u^k = minimiser of I^k, which allows to let $\tau \to 0^+$.

• $$\left\langle \frac{u^k-u^{k-1}}{\tau}, u^k \right\rangle_H = \frac{\tau}{2}\left\| \frac{u^k-u^{k-1}}{\tau} \right\|_H^2 + \frac{1}{2\tau}\left(\|u^k\|_H^2 - \|u^{k-1}\|_H^2 \right). \qquad (0)$$

Discrete Grönwall:

• When $(y^\ell)_{\ell=1,\dots,L} \geq 0$ and $a_0, b_0, b_1, \dots, b_{L-1} \geq 0$ and assume $y^\ell \leq a_0 + \sum_{k=0}^{\ell-1} b_k y^k$, then

$$\max_{\ell \in \{0,\dots,L\}} y^\ell \leq a_0 \exp\left(\sum_{k=0}^{\ell-1} b_k \right)$$

Assumption: Not just I semicoercive, but also its linearisation.

• $\langle DI(v), v \rangle \geq C_1' \|v\|_X^p - C_2 \|v\|_H^2$

• $\|DI(v)\|_{X^*} \leq C_3'(1+\|v\|_X^{p-1})$.

Prop: Let DI be semicoercive, $\tau < \frac{1}{4C_2'}$ then

$$\max_\ell \|u^\ell\|_H + \tau \sum_{k=1}^K \|u^k\|_X^p + \tau \sum_{k=1}^K \left\| \frac{u^k-u^{k-1}}{\tau} \right\|_{X^*}^{p'} + \tau \sum_{k=1}^K \|DI(u^k)\|_{X^*}^{p'} \leq C_0.$$

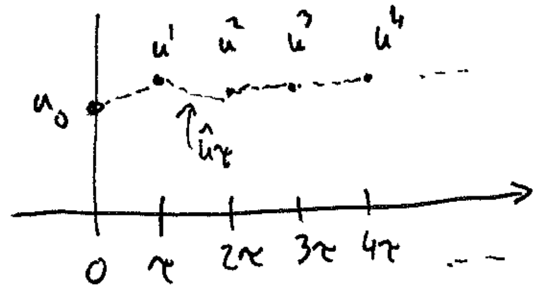
Observations: • u^ℓ bounded in suitable norm $\Rightarrow$ weak cmpt. $\Rightarrow \exists$ conv. subseq.

• $$\underbrace{\left(\frac{u^k-u^{k-1}}{\tau}, u^k \right)_H}_{\overset{(0)}{=}\ \frac{\tau}{2}\left\| \frac{u^k-u^{k-1}}{\tau} \right\|_H^2 + \frac{1}{2\tau}\left(\|u^k\|_H^2 - \|u^{k-1}\|_H^2 \right)} + \underbrace{\langle DI(u^k), u^k \rangle}_{\overset{\text{semicoercive}}{\geq}\ C_1' \|u^k\|_X^p - C_2 \|u^k\|_H^2} = 0 \quad \text{be EL eq. } (**_\tau).$$

$/ \cdot \tau, \sum_k$

• Then use discrete Gronwall.

Do these u^k's converge to a solution of Gradient flow(**)?



Different interpolations:

P.W. constants $\quad u_\tau^+, u_\tau^-$

P.W. linears $\quad \hat{u}_\tau$

$$[0,T] \to H.$$

Hope: $u_\tau^+, u_\tau^-, \hat{u}_\tau \underset{\tau \to 0^+}{\to} u$.

Claim: $\langle \partial_t \hat{u}_\tau, v \rangle + \langle DI(u_\tau^+), v \rangle = 0.$

• ~~Clear by observation~~ that $\partial_t \hat{u}_\tau = \dfrac{u^k - u^{k-1}}{\tau}$ in $((k-1)\tau, k\tau)$.

Translate apriori estimate into estimate for $\hat{u}_\tau, u_\tau^+, u_\tau^-$:

$$\|u_\tau^+\|_{L^\infty(0,T;H)} + \|u_\tau^+\|_{L^p(0,T;X)} + \|\hat{u}_\tau\|_{W^{1,p'}(0,T;X^*)} + \|DI(u_\tau^+)\|_{L^{p'}(0,T;X^*)} \leq C_0.$$

Prop: Assume that $X \underset{cpt.}{\hookrightarrow} H$, then $\exists\, u \in L^p(0,T;X) \cap W^{1,p'}(0,T;X^*)$ and $\xi \in L^{p'}(0,T,X^*)$ and a sequence $\tau_n \underset{n\to\infty}{\to} 0^+$ s.t.

$\hat{u}_{\tau_n} \to u$ (weak*-topology) in $L^\infty(0,T;H)$

$\hat{u}_{\tau_n} \to u$ in $L^p(0,T;X)$

$\hat{u}_{\tau_n} \to u$ in $W^{1,p'}(0,T;X^*)$

$DI(u_{\tau_n}^+) \to \xi$ in $L^{p'}(0,T;X^*)$

Then limit $u \in C(0,T;H)$, $u(0) = u_0$ s.t. satisfies $\langle u', v \rangle + \langle \xi, v \rangle = 0$ $\forall v \in X$. (5)

If $\xi = DI(u) \Rightarrow u$ soln. to gradi~~ent~~ flow.