Variational Methods for PDEs L10

Corollary (to last week's thm)

If $\exists$ infimizing seq. $w_n \in X_n$, then $\|u - u_n\|_X \to 0$.

Definition: Let E be convex & lower semicontinuous.

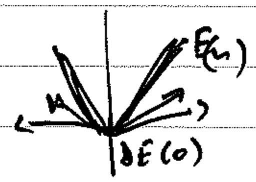
$$\partial E : X \longrightarrow \mathcal{P}(X^*)$$

$$\partial E(u) := \{ v \in X^* : E(w) \geq E(u) + \langle v, w-u \rangle \quad \forall w \in X \}$$

"all directions s.t. directional derivative lies under graph"

When E is C^1 at u, then $\partial E(u) = \{ \nabla E(u) \}$

$\to$ subgradients of u



all slopes $\in [-1, 1]$

Remark: $\cdot$ $0 \in \partial E(u) \implies u$ global minimizer of E.

$\impliedby$ also true if E is sufficiently smooth.

$\cdot$ $\partial E(u) = \{v\} \iff E$ is differentiable (in Gateaux sense)

Proof (thm from lec. 9)

$$\alpha_G \|\Lambda(u-v)\|_Y^2 + \alpha_F \|u-v\|_X^2 \le \frac{1}{2}\left(E(v)+E(u)\right) - E\left(\frac{v+u}{2}\right)$$

$\boxed{\text{strong convexity}}$ with $E(u) = F(u) + G(\Lambda u)$

$$E\left(\frac{u+v}{2}\right)$$

optimality $0 \in \partial E(u)$ $\overset{\shortparallel}{w}$ $\Rightarrow$ $\geq E(u) + \alpha_I \|u - \frac{v+u}{2}\|_X^2$ $\left(\text{use } w=0, \; v \to \frac{v+u}{2}\right)$

$\forall v$ statement $\Rightarrow$ change is ok.

$$\Rightarrow \alpha_G \|\Lambda(u-v)\|_Y^2 + \alpha_F \|u-v\|_X^2$$
$$\le \frac{1}{2}\left(E(u)+E(v)\right) - E(u) - \alpha_I \|u - \frac{v+u}{2}\|_X^2$$
$$= \frac{1}{2}\left(E(u) + E(v)\right) - \frac{\alpha_I}{4}\|u-v\|_X^2$$

Now take $\inf_{v \in X_h}$ $\Longrightarrow$ cor.

□

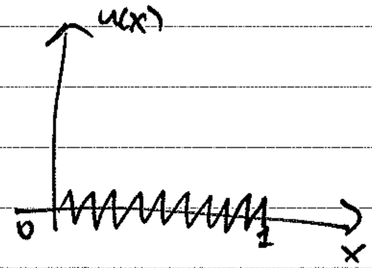
Remark: When $E(u) = \frac{1}{2}\|u\|_H^2 \xrightarrow{-\langle f, u\rangle}$ Cea's lemma

$$E(u) = \frac{1}{2} a(u,u) - \langle f, u\rangle$$

key: α_I, α_F, α_G are properties of E, not X_h.

- In general, without strong convexity assumptions, there is no reason to expect convergence in norm.
- without convexity: Some properties might still be approximated by u_h, even if $u_h \not\to u$.

For example ;



$$E(u) = \int_0^1 (|\partial_x u|^2 - 1)^2 + u^4$$

on $X = W_0^{1,4}(0,1)$

An infimizing sequence consists of functions of slope ± 1, with decreasing ~~...~~ absolute value of ~~.~~ u. Half of the time it must have slope $+1$, and half -1.

Young measure
$$\nabla u = \frac{1}{2}(\delta_{-1} + \delta_{+1})$$
Probability distribution in every point X.

Numerical computations can approximate this ~~young~~ Young measure
$\Uparrow$ ——— $\|u\|_{L^4} = 0$

Remark : Other error estimates

a priori error estimate → good for qualitative conclusions

"Cea's lemma": $\alpha \|u - u_h\|_X^2 \leq \frac{1}{2}(E(u_h) - E(u)) \ \forall u_h \in X_h$
(our Thm for $G = 0$)

FEM error is best possible error (up to constant multiples)

Given u_h, we still have no idea how big rhs is. We only know, for example, that rhs $\to 0$ with a certain rate as $h \to 0$.

Theorem in Bartels, part b: $\|u - u_h\|_X^2 \leq$ (computable from u_h)

a posteriori error estimates → good for devising fast methods.

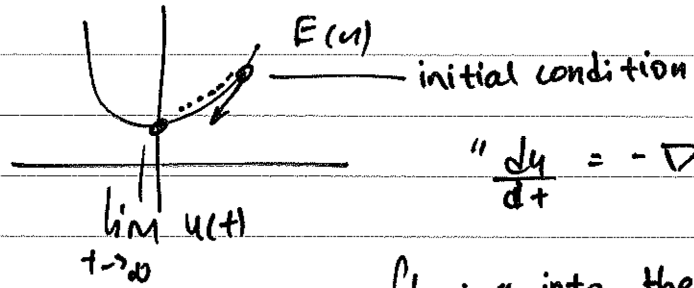
This gives you a computable upper bound of error, once you've calculated your solution, u_h. This allows you to develop efficient algorithms, by investing efforts where error is large.

__So far__: Stationary minimization problem:

$$E: X \to \mathbb{R} \cup \{+\infty\}$$

- Does there exist a minimizer?
- Can we approximate it?

__Now__: Dynamic problems ('heat equations')



$$\text{"} \frac{du}{dt} = -\nabla_u E(u) \text{"}$$

flowing into the direction of steepest descent.

__Hope__: $\lim_{t\to\infty} u(t)$ gives (local or global) minimizer.

__Basic questions__:
- Make sense of $\frac{du}{dt} = -\nabla_u E(u)$
  - $\to \nabla_u E(u)$
  - $\to$ Do there exist solutions for given IC?
- long-time behaviour?
- use this for interesting applications?

__Basic idea in finite dimensions:__

$y(t) \in \mathbb{R}^n$, $E: \mathbb{R}^n \longrightarrow \mathbb{R}$

$$y'(t) = -\nabla E(y(t)), \quad y(t=0) = y_0$$

If E is C^2 or even just DE Lipschitz
$\Rightarrow \exists!$ solution (Picard-Lindelöf) for short times.

Take scalar product of $\boxed{*}$ with $y(t)$:

chain rule

$$\underbrace{\langle y', y' \rangle}_{\|y'\|^2} = -\langle y', DE(y) \rangle = \frac{-d}{dt} E(y(t))$$

integrate: $\displaystyle\int_0^T \|y'\|^2 dt = \int_0^T \frac{-d}{dt} E(y(t))\, dt$

$$\Rightarrow -E(y(t)) + E(y(0)) = \int_0^T \|y'\|^2 dt$$

$$\Rightarrow E(y(t)) = E(y(0)) + \int_0^T \|y'\|^2 dt$$

$\Rightarrow E(y(t)) \leq E(y(0))$ i.e. $t \longmapsto E(y(t))$ is a decreasing function

It becomes stationary if $DE(y) = 0$
$\Rightarrow$ stationary points are critical points of E.

In ∞-dimension, we need appropriate function spaces of time-dependent functions s.t. a solution exists and the PDE has a meaning.

Def.: $L^p([0,T], X) = \{ u : [0,T] \to X$ "weakly measurable" $\forall \varphi \in X^*$, $\langle \varphi, u \rangle$ measurable;

$$\underbrace{\left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p}}_{\|u\|_{L^p([0,T],X)}} < \infty \}$$

This is a Banach space for $1 \leq p \leq \infty$. Hilbert space for $p = 2$ if X is a Hilbert space.

Def.: We say $w \in L^1([0,T],X)$ is the weak derivative of $u \in L^1([0,T],X]$ if

$$\int_0^T \varphi'(t)\, u(t)\, dt = -\int_0^T \varphi(t)\, w(t) \qquad \forall \varphi \in C_c^\infty([0,T])$$

Note: $\varphi \in C_c^\infty(0,T)$ is a scalar function.

Def.: $W^{1,p}([0,T],X) = \{ v \in L^p([0,T]X) : v' \in L^p([0,T],X) \}$

norm: $\|u\|_{W^{1,p}([0,T],X)} := \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p \right)^{1/p}$

Banach spaces for $p \in [1,\infty]$, Hilbert for $p=2$ with X Hilbert.

Basic fact: Sobolev embedding theorem in 1d:

$$W^{1,p}([0,T]X) \hookrightarrow C([0,T],X) \quad \text{i.e.}$$

every function in $W^{1,p}([0,T],X)$ has a continuous rep/ is continuous.

Furthermore $\|u\|_{C([0,T]X)} := \max_{[0,T]} \|u(t)\|_X$

$$\leq C \|u\|_{W^{1,p}}$$

Def.: H separable Hilbert space, $H \simeq H'$

$X \hookrightarrow H \hookrightarrow X^*$ "evolution triple."
densely, continuously

Basic fact:

$$\langle u(t_2), v(t_2) \rangle_H - \langle u(t_1), v(t_1) \rangle_H = \int_{t_1}^{t_2} \langle u'(t), v(t) \rangle + \langle v'(t), u(t) \rangle \, dt$$

for $u \in L^p([0,T],X)$, $u' \in L^{p'}([0,T],X^*)$ (p': $\frac{1}{p} + \frac{1}{p'} = 1$)
i.e. can integrate by parts.

$\hookrightarrow$ Energy estimates