From last week: Γ-convergence

Variational Methods 12.2.19 (1)

Special case: $W(\nabla u)$ quadratic

$$E(u) = \frac{1}{2} a(u,u) - \langle f, u \rangle \quad \text{on Hilbert space}$$

- Lax Milgram for quadratic problems

Prop 4.1: $E_h \xrightarrow{\Gamma} E \Rightarrow E$ is w.l.s.c.

Proof: $(u_j) \subseteq X$ seq. $u_j \xrightarrow{\omega} u$ $\quad E(u_j)$

$u_j^h \in X_h$ $\quad E_h(u_j^h)$

Γ-conv

$\forall j$ choose h_j s.t. $|E(u_j) - E_{h_j}(u_j^{h_j})| \leq \frac{1}{j}$

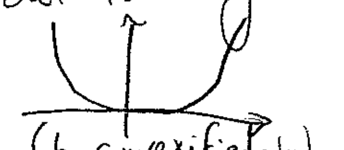
If $h_j \xrightarrow{j \to \infty} 0$. Then $u_j^{h_j} \xrightarrow{\omega} u$

$E(u) \leq \liminf E_{h_j}(u_j^{h_j}) = \liminf E(u_j)$. #

Upcoming: $W =$  $\rightarrow E(u) = \int_\Omega W(\nabla u)\,dx$

Discretise this with p.w. linears.

$\Rightarrow E_h(u_h) \xrightarrow{\Gamma} ?$ (Yes.) but to wrong soln.



(to convexified soln).

Two more facts: • $E_h \xrightarrow{\Gamma} E$, G w-cont. $\Rightarrow E_h + G \xrightarrow{\Gamma} E + G$.

(True, as $G(u_h) \to G(u)$ by cont.)

• $E_h \xrightarrow{\Gamma} E$ and $u_h \in X$: $I_h(u_h) \leq \inf_{v_h \in X_h} I_h(v_h) + \varepsilon_h$, $\varepsilon_h \to 0$ and if $u_h \xrightarrow{\omega} u$, then $E_h(u_h) \to E(u)$ and u minimizes E.

② <u>Message</u>: If u_h is minimizer of E_h over $X_h \subset X$ and $E_h \xrightarrow{\Gamma} E$ and if u_h converges, then u_h converges to minimizer of E.

<u>Ex</u>: Let $X = H_0^1(\Omega)$, $X_h =$ p.w. linears on $\mathcal{T}_h$.

$$E(u) = \frac{1}{2}\int_\Omega (\nabla u)^2 - \int_\Omega f u$$

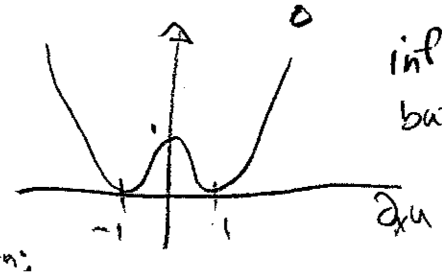
$$E_h(u_h) = \frac{1}{2}\int_\Omega (\nabla u_h)^2 - \int_\Omega f u_h = E(u_h)$$

Γ by continuity. ($f \in L^2$).

$$\int_\Omega (\nabla u_h)^2 \xrightarrow{\Gamma} \int_\Omega (\nabla u)^2.$$

So $E_h(u_h) \xrightarrow{\Gamma} E(u)$.

<u>Ex</u>: Let $E(u) = \int_0^1 ((\partial_x u)^2 - 1)^2 + u^4 \, dx$ on $X = W^{1,4}(0,1)$.

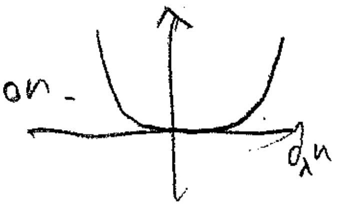


$\inf E = 0$
but min $\nexists$

Assertion:
$E(u)$ discretized on $X_h =$ p.w. linears on uniform mesh of $[0,1]$.

$E_h(u_h) := E(u_h) \quad \forall u_h \in X_h$.

$\Rightarrow E_h \to E^{**} =$ convexification.



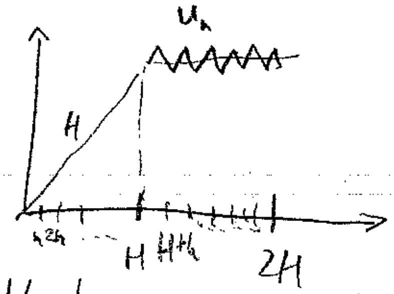
Γ-conv.: $E_h \xrightarrow{\Gamma} E$ as $E_h(u_h) = E(u_h) \geq E^{**}(u_h)$

So, $\liminf_h E_h(u_h) \geq \liminf_h E^{**}(u_h) \geq E^{**}(u)$.
convex

- We want $\forall u \in X$, a sequence $u_h \xrightarrow{w} u$ s.t. energies converge.

We do this first for $u \in X_H$

$$E(u_h) \simeq E^{**}(u)$$



In general, for $u \in X$: Choose H s.t. $u - u_H$ small enough

$u_H \leftarrow u_h$ + triangle ineq.

Goal: "Cea's" lemma for nonlinear functionals.

$$\|\text{num. error}\| \leq C \|\text{best possible error}\|.$$

Method: FEM for nonlinear problems

Assume: $\cdot\, E : X \to \mathbb{R} \cup \{+\infty\}$, assume $\exists$ unique minimizer $u \in X$.
$\qquad\quad \cup\!\mid$
$\cdot\ X_h$ fin. dim.

Compute minimizer u_h of E restricted to X_h.

Thm: $(\ldots) + \left(\alpha_F + \frac{\alpha_I}{4}\right) \|u - u_h\|_X^2 \leq \frac{1}{2} \inf_{w_h \in X_h} \left(E(w_h) - E(u)\right)$

$\leq$ Error of ground state Energy (best possible).

(4.) Let X, Y banach $\Lambda : X \to Y$ bounded, linear.

$$F : X \to \mathbb{R} \cup \{+\infty\}, \quad G : Y \to \mathbb{R} \cup \{+\infty\}$$

convex, l.s.c, proper functionals.

Wanted: Minimize $E(u) = F(u) + G(\Lambda u)$

Set $F^* : X' \to \mathbb{R} \cup \{+\infty\}$ Fenchel conjugate

$$F^*(w) = \sup_{v \in X} \langle w, v \rangle - F(v)$$

$F^{**}(w) = F(w)$ in reflexive Banach.
In general convexification.

Similarly for G.

For "Cea's" lemma (no need for conjugate).
Assume F or G is strongly convex:

$$\alpha_F \|v_1 - v_2\|^2 + F\left(\frac{v_1+v_2}{2}\right) \le \frac{F(v_1) + F(v_2)}{2} \quad (\text{same for } G)$$

Then, $\frac{1}{2}(E(v) + E(u)) - E\left(\frac{v+u}{2}\right) = \frac{1}{2} F(v) + \frac{1}{2} G(\Lambda v) + \frac{1}{2} F(u) + \frac{1}{2} G(\Lambda u) - F\left(\frac{v+u}{2}\right) - G\left(\Lambda \frac{v+u}{2}\right)$.

$\ge \alpha_F \|u-v\|^2 + \alpha_G \|\Lambda(u-v)\|^2$

As u is minimizer $\Rightarrow E($