Past weeks: X = Sobolev space

$$E: M \subseteq X \to \mathbb{R} \cup \{+\infty\}, \quad E(u) = \int_\Omega W(x, u, \nabla u)\,dx$$

Does minimizer exist?

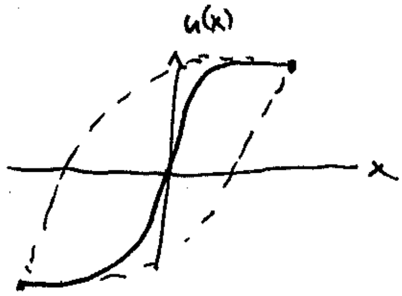
Motivation: Min/stationary states/points satisfying PDE (E-L eqns.) under some assumptions.

Next few lectures: Compute/approximate minimizers.

Recall: Lavrentier phenomenon (Ex 2.4 in JB notes)

$$E(u) = \int_{-1}^{1} (x^3 - u^5)^2 (\partial_x u)^{20} + \varepsilon (\partial_x u)^2 \, dx, \quad 0 < \varepsilon \ll 1.$$

- $u(x) = \left(x^3\right)^{\frac{1}{5}}$



- $\min\limits_{u \in W^{1,p}} E(u) < \min\limits_{u \in W^{1,\infty}} E(u)$.

Standard numerical methods: Approximate u by "nice" functions

e.g. p.w. polynomials on a mesh $x_0 = -1 < x_1 < x_2 \ldots < x_n = 1$.

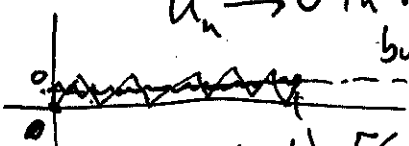
Then usually $u_h \in W^{1,\infty}$.

This will <u>not</u> converge to minimizer in $W^{1,p}$.

Problems:
- Might converge to wrong "solution" (...).
- Might converge, but no solution exists.

$$E(u) = \int_0^1 \left(((\partial_x u)^2 - 1)^2 + u^4 \right) dx \quad \text{on } W^{1,4}(0,1).$$

$\inf E(u) = 0$.



$u_n \to 0$ in $W^{1,4}$ but $\partial_x u = \pm 1$ a.e.

So $E(u_n) \to 0$ as $h \to 0$, but $E(0) = 1$.

There is no minimizer. Since $\nexists (u=0$ with $\partial_x u = \pm 1)$. (2.)

Question: • Under which conditions will $u_h \to u$?
• If it converges, what are the convergence rates; In what sense does it converge?

Analytically: • Stronger notions of convergence than weak convergence.
• Quantitative estimates in h rather than "$\exists$ limits".

Review: . The basic theory of FEM

∘ Linear PDEs $\Leftrightarrow$ quadratic functionals $E(u)$.

• Let H be Hilbert space (think of $W^{1,2}(\Omega)$).

• bilinear form $a: H \times H \to \mathbb{R}$, continuous, coercive

$|a(u,v)| \lesssim \|u\|_H \|v\|_H \quad \|u\|_H^2 \lesssim a(u,u)$

$\forall u,v \in H$.

∘ If $a(\cdot,\cdot)$ symmetric, then $|||u||| := a(u,u)^{\frac{1}{2}}$ is equivalent norm on H by continuity & coercivity.

Thm: (Lax-Milgram) If $f: H \to \mathbb{R}$ is cont. linear functional and $a(\cdot,\cdot): H \times H \to \mathbb{R}$ is symmetric, coercive, & continuous bilinear form on H, then

$$E(u) = \frac{1}{2} a(u,u) - \langle f, u \rangle \text{ has a minimizer.}$$

∘ This is a special case of thm. from W2:

- $E(u) \to \infty$ as $u \to \infty$ by coercivity.

• $E(u)$ is w.l.s.c. as $a(u,u) \simeq \|u\|_H^2$.

Now: Approximate minimizers.

- $u \in H$ minimizer of $E(u) = \frac{1}{2} a(u,u) - \langle f, u\rangle$.
- $H_h \subseteq H$ finite dim. ~~minimizer~~ subspace
- $u_h \in H_h$ minimizer of E restricted to H_h. (LM $\Rightarrow \exists!$).

Question: Does $u_h \to u$ if $\overline{\bigcup_{h>0} H_h} = H$?

- If so, in what sense?

Thm: (Cea's lemma)

$$\|u - u_h\|_H \leq \tilde{C} \inf_{v_h \in H_h} \|u - v_h\|_H .$$

Pf.:
$$\|u - u_h\|_H^2 \leq \frac{1}{\alpha} a(u - u_h, u - u_h)$$
$$= \frac{1}{\alpha}\left[a(u, u-u_h) - a(u_h, u - u_h)\right]$$
$$(\text{by orthogonality}) = \frac{1}{\alpha}\left[a(u - v_h, u - u_h)\right] \qquad \forall v_h \in H_h.$$
$$\leq \frac{C}{\alpha} \|u - v_h\|_H \|u - u_h\|_H$$
$$\leadsto \|u - u_h\|_H \leq \frac{C}{\alpha} \inf_{v_h} \|u - v_h\|_H \quad \text{as } v_h \text{ is arbitrary.}$$

- On computer for Laplace eq.:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad \Longleftrightarrow \min E(u) = \frac{1}{2}\int_\Omega (\nabla u)^2 - \langle f, u\rangle \text{ over } W_0^{1,2}(\Omega).$$

Appox of u: H_h = p.w. linear polys. on triangulation of Ω.

$\Rightarrow$ Linear algebra problem $a(u_h, v_h) = \langle f, v_h\rangle \ \forall v_h \in H_h$

Gamma-convergence of discretisations

- Min $E(u)$ on $\underset{\cup}{X}$. ← Cont. problem

Min $E_h(u_h)$ on X_h. ← Approximation.

Before break: $E(u) = \frac{1}{2} a(u,u) - \langle f,u \rangle$ on $\underset{\cup}{H}$

$E_h(u_h) = E(u_h)$ on H_h

Now: X Banach space

E nonlinear (often non-quadratic) functional

Extend E_h to X by setting $E_h(u) = \begin{cases} E_h(u) & \text{if } u \in X_h \\ +\infty & \text{if } u \notin X_h \end{cases}$

Def: $\{E_h : X \to \mathbb{R} \cup \{+\infty\}\}_{h>0} \xrightarrow{\Gamma} E$ as $h \to 0^+$.

w.r.t. fixed topology ω on X if the following hold:

(I) For every $\{u_h\}_{h>0} \subseteq X$ with $u_h \xrightarrow{\omega} u$ for some $u \in X$ we have $\liminf_{h \to 0^+} E_h(u_h) \geq E(u)$

(II) For every $u \in X$ $\exists (u_h)_{h>0} \subseteq X$ with $u_h \xrightarrow{\omega} u$ s.t. $E_h(u_h) \to E(u)$ as $h \to 0^+$.

Implications: From (I) E is lower bound for sequence E_h.

From (II) lower bound is attained.

- ω is always going to be the weak topology for us, unless otherwise stated.

Thm: Assume discretisation is conforming and that $\overline{\bigcup_{h>0} X_h} = X$. Assume that E is wlsc and continuous in $\|\cdot\|$.

Then $E_h \xrightarrow{\Gamma} E$ as $h \to 0^+$.

Pf.: · Let $(u_h)_{h>0} \subset X$ s.t. $u_h \rightharpoonup u$ as $h \to 0^+$.

WTS: $\liminf_h E_h(u_h) \geq E(u)$

$\overset{\text{by}}{\underset{\text{conformity}}{\Leftrightarrow}}$ $\liminf_h E(u_h) \geq E(u)$

As E is wlsc it's clear. ✓(i)

· WTS: $\exists$ of $(u_h)_{h>0}$: $E_h(u_h) \to E(u)$.

By assumption of 'closure', $\exists (u_h \in X_h)_{h>0}$ s.t. $u_h \to u$ in norm

· $E_h(u_h) = E(u_h) \xrightarrow[\text{norm cont.}]{} E(u)$. ✓(ii)

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