Variational Methods L 7

Thm (John Ball) $W : \mathbb{R}^{3\times 3} \to [0,\infty]$ $\mathrm{adj}\,\phi = \det\phi\,\phi^{-1}$

$W(\phi) = g(\phi, \mathrm{adj}\,\phi, \det\phi)$, g convex

Let $\Omega \subset \mathbb{R}^3$,

$u_m, u \in H^{1,3}_{loc}(\Omega, \mathbb{R}^3)$. Suppose

- $u_m \rightharpoonup u$ ~~weakly~~ in $H^{1,3}(\Omega', \mathbb{R}^3)$ $\forall \Omega' \subset\subset \Omega$
- $\det(\nabla u_m) \rightharpoonup h$ in $L^1(\Omega')$, $h \in L^1_{loc}(\Omega)$

$$\Longrightarrow \int_\Omega W(\nabla u)\,dx \le \liminf_{m\to\infty} \int_\Omega W(\nabla u_m)\,dx.$$

Applications mentioned in last lec $\to \phi$ deformation, compensated compactness.

Proof: Show that $\mathrm{adj}(\nabla u_m) \rightharpoonup \mathrm{adj}(\nabla u)$ in $L^1(\Omega')$ $u \in W^{1,3}(\Omega) \Rightarrow \nabla u \in L^3(\Omega)$

Same for det.

Then copy the proof for convex integrands.

Weak cgce. $\varphi_m \rightharpoonup \varphi \iff \langle \varphi_m, \phi\rangle \to \langle \varphi, \phi\rangle \quad \forall \phi$

Here $\det(\nabla u_m)$ is highly nonlinear.

First the simpler case: $\boxed{\mathrm{adj}\,\nabla u_m}$ # entries are mod 3.

$L^{3/2}(\Omega) \ni \mathrm{adj}\,\nabla u_m$ is a matrix with entries $\dfrac{\partial u_m^{i+1}}{\partial x^{j+1}} \dfrac{\partial u_m^{i+2}}{\partial x^{j+2}} - \dfrac{\partial u_m^{i+2}}{\partial x^{j+1}} \dfrac{\partial u_m^{i+1}}{\partial x^{j+2}}$

$\exists$ weakly cgt subseq

$$= \partial_{x^{j+1}}\left(u_m^{i+1} \frac{\partial u^{i+1}}{\partial x^{j+2}}\right) - \partial_{x^{j+2}}\left(u_m^{i+1} \frac{\partial u_m^{i+2}}{\partial x^{j+1}}\right)$$

$u_m \in H^{1,3} \xrightarrow[\text{Rellich}]{\text{cpt}} L^3 \quad \Rightarrow \quad u_m \rightharpoonup u$ in $H^{1,3}$, $u_m \to u$ in L^3

These these cge weakly since they are of the form $u_m \nabla u_m$

Lemma: $u_m \nabla u_m \rightharpoonup u \nabla u$ in $L^{3/2}$

Proof: $\forall \varphi \in (L^{3/2})^*$: $\langle u_m \nabla u_m, \varphi\rangle \to \langle u \nabla u, \varphi\rangle$

$= \langle u_m \nabla u_m - u\nabla u_m + u \nabla u_m, \varphi\rangle$

$= \langle (u_m - u)\nabla u_m + u \nabla u_m, \varphi\rangle$

$= \underbrace{\langle (u_m - u)\nabla u_m, \varphi\rangle}_{\to 0} + \langle u \nabla u_m, \varphi u\rangle \to \langle \nabla u, \varphi u\rangle$

$= \langle u \nabla u, \varphi\rangle$

Lemma: $\varphi_m \to \varphi$ in L^p $\Rightarrow$ $\partial x_j \varphi_m \to \partial x_j \varphi$ in the sense of distributions

Proof: WTS $\langle \partial x_j \varphi_m, \psi \rangle \to \langle \partial x_j \varphi, \psi \rangle$ $\forall \psi \in C_c^\infty(\Omega)$

$$\langle \partial x_j \varphi_m, \psi \rangle = \langle \varphi_m, \underbrace{\partial x_j \psi}_{\in C_c^\infty(\Omega)} \rangle \qquad \langle \partial x_j \varphi, \psi \rangle = \langle \varphi, \partial x_j \psi \rangle$$

To show: $\langle \varphi_m - \varphi, \partial x_j \psi \rangle \to 0$ But this since $\varphi_m \to \varphi$

$$\langle \varphi_m - \varphi, \partial x_j \psi \rangle \leq 0 \leftarrow \|\varphi_m - \varphi\|_{L^p} \underbrace{\|\partial x_j \psi\|_{(L^p)^*}}_{<\infty}$$

"Taking derivatives does not destroy week convergence"

Conclusion: adj ∇u_m cges in sense of distributions.

There also exists a weakly cgt subseq in $L^{3/2}(\Omega)$ [see above]. By uniqueness of limits adj$(\nabla u_m) \to$ adj(∇u).

It remains to show that $\det(\nabla u_m) \to \det(\nabla u)$ in $L^1(\Omega')$

Algebraic identity (expand det along 1st row)

$$\det(\nabla u_m) = \frac{\partial u_m^1}{\partial x^1}\left(\frac{\partial u_m^2}{\partial x_2}\frac{\partial u_m^3}{\partial x_3} - \frac{\partial u_m^2}{\partial x_3}\frac{\partial u_m^3}{\partial x_2}\right) - \frac{\partial u_m^1}{\partial x_2}\left(\cdots \right) + \frac{\partial u_m^1}{\partial x_3}\left(\cdots \right)$$

$$= \frac{\partial}{\partial x^1}\left(u_m^1\left[\frac{\partial u_m^2}{\partial x_2}\frac{\partial u_m^2}{\partial x^3} - \frac{\partial u_m^3}{\partial x_2}\frac{\partial u_m^2}{\partial x_3}\right]\right) - \frac{\partial}{\partial x^2}\left(u_m^1(\cdots)\right) - \frac{\partial}{\partial x^3}\left(u_m^1(\cdots)\right)$$

$u_m^1 \xrightarrow[\text{in } L^3]{} u^1$

$(\cdots) \longrightarrow$ in $L^{3/2}(\Omega')$

- From Lemma, $u_{m}^{i}[\ ] \to$

- $\frac{\partial}{\partial x_j}(u_m^i[\ ])$ ~~~~ cges in the sense of distributions
  This limit needs to agree with L^1-limit by uniqueness of the limit.

Thm $\Longrightarrow$ $E(u) = \int_\Omega W(\nabla u)\,dx$ has minimizer.

uniqueness is far from clear.

Conclusion: This establishes $\exists$ce of solutions to many eqns in non-lin cont mech.
But Not uniqueness.

Compensated Compactness Lemma

Special case of compensated compactness.

Div-curl lemma: $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ in $L^2(\Omega, \mathbb{R}^3)$, $\Omega \subset \mathbb{R}^3$

- div u_m has convergent subseq in $H^{-1}(\Omega)$
- curl v_m $\underbrace{\text{———— '' ————}}_{\text{"relatively compact" in } H^{-1}}$

$\Longrightarrow \forall \varphi \in C_0^\infty(\Omega)$

$$\int_\Omega u_m \cdot v_m \,\varphi\, dx \longrightarrow \int_\Omega u \cdot v \,\varphi\, dx$$

Colloquially: u_m, v_m weakly cgt $\Rightarrow$ $u_m v_m$ cges as distribution if we know something about derivatives of u_m, v_m

__Lemma__: $K \subseteq L^2((0,2\pi)^n)$ rel. compact

$$\Longleftrightarrow \sum_{|\alpha| \geq \alpha_0} |\mu_f(\alpha)|^2 \longrightarrow 0 \quad \forall\ \alpha_0 \to \infty \quad \forall f \in K$$

$\boxed{\text{Pf Exercise!}}$ where $\mu_f(\alpha)$ are the Fourier coeffs. of f.

A statement for H^{-1} follows from this:

$\mu_f(\alpha) \longrightarrow \frac{\mu_f(\alpha)|\alpha|}{1+|\alpha|^2}$ isomorphism $(L^2)^\wedge \to (H^{-1})^\wedge$ $\quad \{(X)^\wedge$:= fourier coeffs of X functions

__Proof__: $f \in L^2 \underset{\text{Parseval}}{\Longleftrightarrow} \sum_\alpha |\mu_f(\alpha)|^2 < \infty$

$$f \in H^1 \Longleftrightarrow \sum_\alpha \left(\frac{|\alpha|}{1+|\alpha|^2}\right)^2 |\mu_f(\alpha)|^2 < \infty$$