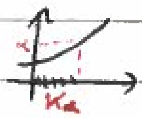
Var methods 2

Last week: • M top. (Hausdorff) space



• $E: M \to \mathbb{R} \cup \{+\infty\}$

• $\boxed{\forall \alpha \in \mathbb{R}: K_\alpha = \{u \in M : E(u) \leq \alpha\} \text{ compact}}$

$\Rightarrow$ E weakly lower semi continuous $u_m \to u \Rightarrow E(u) \leq \liminf E(u_m)$

$\Rightarrow$ E bounded from below, $\boxed{\exists \text{ minimizer}}$

• V reflexive Banach space $W^{k,p}(\Omega)$, $p \in (1,\infty)$
• $M \subseteq V$ weakly closed
• $E: M \to \mathbb{R} \cup \{+\infty\}$
  • coercive: $E(u) \to \infty$ as $\|u\|_V \to \infty$
  • weakly lower semi continuous:
    $\forall \{u_m\}$ with $u_m \rightharpoonup u$: $E(u) \leq \liminf E(u_m)$

$\Rightarrow$ E bounded below, $\boxed{\exists \text{ minimizer.}}$

Generalises Lax-Milgram

Sobolev embedding theorem: $\Omega \subset \mathbb{R}^n$ bounded

$\partial\Omega$ Lipschitz.

If $kp < n$, then $\boxed{W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)}$ $\forall 1 \leq q \leq \frac{np}{n-kp}$

$\boxed{W^{k,p} \hookrightarrow L^q(\Omega) \text{ is compact if } q < \frac{np}{n-kp}}$

~~Basic example: $W^{1,2}(\Omega)$, $\Omega = (a,b)$ $n=1$~~

$W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ $\qquad 1 \le q \le \frac{2n}{n-2} > 2$

$n \ge 3$

$n=3$: $\qquad W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$

Today: Which functionals E satisfy wlsc

i.e. $u_m \rightharpoonup u \Rightarrow E(u) \le \liminf_{m\to\infty} E(u_m)$

First: $E(u) = \int_\Omega F(x, u(x), \nabla u(x))\, dx$, $\quad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$

bounded $\subset \mathbb{R}^n$
Lipschitz boundary

Definition: $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is Caratheodory

if
- $x \mapsto F(x,u,p)$ measurable $\forall u,p$ (fixed),
- $(u,p) \mapsto F(x,u,p)$ continuous $\forall x$.

Var methods 2

(1.6) Theorem: • F Caratheodory

• $F(x,u,p) \geq \phi(x)$ for a.e. x,u,p for some $\phi \in L^1(\Omega)$

$\inf E > -\infty$

"ellipticity" of PDE $\Leftarrow$ • $p \mapsto F(x,u,p)$ convex for a.e. x,u

$\Rightarrow$ If $u_m, u \in W^{1,1}_{loc}(\Omega)$ and $u_m \to u$ in $L^1(\Omega')$

$\nabla u_m \rightharpoonup \nabla u$ in $L^1(\Omega')$, then

$\forall \Omega' \subset\subset \Omega$

$$\boxed{E(u) \leq \liminf_{m\to\infty} E(u_m)}$$

Colloquially: F convex $\Rightarrow$ E wlsc.

Example: • $E(u) = \frac{1}{2}\int_\Omega (\nabla u)^2\,dx$

$$F(x,u,p) = \frac{1}{2}|p|^2 \qquad \text{convex}$$

• $E(u) = \frac{1}{q}\int_\Omega |\nabla u|^q\,dx - \int_\Omega f u\,dx, \qquad f \in L^1(\Omega)$

$$F(x,u,p) = \frac{1}{q}|p|^q - f(x)\,u$$

Long History: Morrey, Tonelli, ...

Proof: $F(x,u,p) = \tilde{F}(x,p)$, for simplicity.

General case: see Struwe's book

By assumption, $\inf E > -\infty$. So after passing to a subsequence, again called u_m, we may assume $E(u_m)$ finite, $\geq C$ and $E(u_m)$ converges.

After replacing ~~F~~ F by $F-\Phi$, we may assume $F \geq 0$.

Fix $\Omega' \subset\subset \Omega$.

<u>Recall:</u> Mazur's lemma:

$$x_n \rightharpoonup x \implies \exists\, y_n = \sum_{k=1}^{\hat{n}} \alpha_k^{\hat{n}} x_k, \quad \sum_n \alpha_n^{\hat{n}} = 1, \ \alpha_k^{\hat{n}} \geq 0$$

i.e. convex combination

s.t. $\|y_n - x\| \to 0$

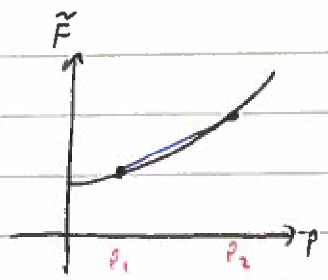
Slight refinement in measure theory: $L^1(\Omega')$

- $\|y_n - x\|_{L^1(\Omega')} \to 0$
- $y_n \to x$ in almost every point.

<u>For us:</u> $\nabla u_m \rightharpoonup \nabla u \implies p^l \to \nabla u$ in $L^1(\Omega')$

and $p^l = \sum_{k=m_0}^{l} \alpha_k^l \nabla u_k$ convex combination

<u>Convexity:</u> $\tilde{F}(x,p) \leq \sum_{k=m_0}^{l} \alpha_k^l \tilde{F}(x, p_k)$ linear interpolation



if $p = \sum \alpha_k^l p_k$

$p = p^l$

$p_k = \nabla u_k$

Fatou's lemma

$$\int_{\Omega'} F(x, \nabla u(x))\, dx \leq \liminf_{l\to\infty} \int_{\Omega'} \tilde{F}(x, p^l(x))\, dx$$

$$\leq \sup_{m \geq m_0} \int_{\Omega'} \tilde{F}(x, \nabla u_m)$$

(m_0 arbitrary) $$= \limsup_{m\to\infty} \int_{\Omega'} \tilde{F}(x, \nabla u_m)$$

Use that $E(u_m)$ converges to conclude (after some technicalities) that

$$\limsup_{m\to\infty} \int_\Omega = \liminf_{m\to\infty} \int_\Omega \qquad \square$$

To sum up the proof : • "$\rightharpoonup$" $\Rightarrow$ norm convergence of convex combinations

• F convex $\Rightarrow$ "$\leq$"

Remark : Theorem generalizes to

$$E(u) = \int_\Omega F(x, \underbrace{u, \nabla u, \nabla^2 u, \dots,}_{=:\, U} \underbrace{\nabla^m u}_{\text{contains } \nabla U})\, dx$$

In fact, this is a special case with U as above.

Example : $E(u) = \frac{1}{2} \int_\Omega (\Delta u)^2\, dx$ biharmonic problems
plates

PDE : $\boxed{\Delta^2 u = 0}$

Remark : To show $\nabla u_m \to \nabla u$ for minimizing sequence for ex.

$$F(x, z, p) \geq |p|^\mu - \phi(x)$$

("coercivity") More later.

Question : Is $E(u) = \int_\Omega F(x, u, \nabla u)\, dx$ wlsc

"$\iff$" F convex in $\nabla u = p$?

Answer: 1d : Yes $\underset{\Downarrow \text{ convexity}}{}$ Legendre - Hadamard condition $\underset{\Uparrow \text{ existence of minimizers}}{}$

higher dim : No. Generalized convexity conditions

Poly convexity $\leq$ rank - convexity

$\leq$ quasi-convexity

$\Updownarrow$

existence of minimizers

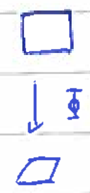
(poly convexity)

Theorem : $W(\Phi) = g(\Phi, \operatorname{adj} \Phi, \det \Phi)$ where $\Phi \in \mathbb{R}^{3\times 3}$,

(John Ball, 70s)

$\operatorname{adj}(\Phi) := \det(\Phi)\, \Phi^{-1}$ (if Φ^{-1} exists)

Note : Cramer's rule : $A^{-1} = \frac{1}{\det A} \begin{pmatrix} \begin{vmatrix} (n-1)\times(n-1) \\ \text{matrix} \end{vmatrix} & |\cdot| & |\cdot| \\ |\cdot| & |\cdot| & |\cdot| \\ |\cdot| & |\cdot| & |\cdot| \end{pmatrix}$


Φ

Deformation

$\det \Phi$

~ compression/ stretching of material

and g convex, ≥ 0

Let $\Omega \subset \mathbb{R}^3$, $\cancel{u_m, u \in W^{1,3}(\Omega; \mathbb{R}^3)}$

$u_m, u \in W^{1,3}(\Omega; \mathbb{R}^3)$

- $u_m \rightharpoonup u$ in $W^{1,3}(\Omega; \mathbb{R}^3)$ $\forall \Omega' \subset\subset \Omega$
- $\det(\nabla u_m) \rightharpoonup h$ in $L^1(\Omega')$, where $h \in L^1_{loc}(\Omega)$

$\Rightarrow \int_\Omega W(\nabla u)\, dx \leq \liminf_{m\to\infty} \int_\Omega W(\nabla u_m)\, dx$

- Application: For poly convex materials, there exist deformations Φ which minimize the energy

Var methods

- Techniques behind it: Compensated compactness

Convergence of $F(u_n)$ if $u_n \to u$.
and F nonlinear functional. without convexity.