1

Variational Methods: Lecture 1 (with Heiko)

Recall: • $W^{1,p}(\Omega) = \{ f \in L^p(\Omega) : \partial_x f \in L^p(\Omega) \}$
we could consider $\Omega \subseteq \mathbb{R}^n$ (usually bounded).

• functionals $E(u) = \int_\Omega F(x, u, \nabla u)\, dx$
on a suitable topological space $\ni u$

Previously we have seen a discussion of minimizers of E

E.g • To find equilibrium/ground state of physical system
• Find a minimal surface = surface of minimal area given boundary
• Find geodesics
• Find best approximation of data by a function

Soap bubbles

Length minimising curve

General Set up I:

Consider V a Banach Space, the dual space V^* and $E : V \to \mathbb{R}$ a functional

Assume Fréchet differentiable

DE derivative: $\langle v, DE(u) \rangle := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u + \varepsilon v)$
$V \to V^*$

directional derivative in direction $v \in V$
(Gateaux derivative)

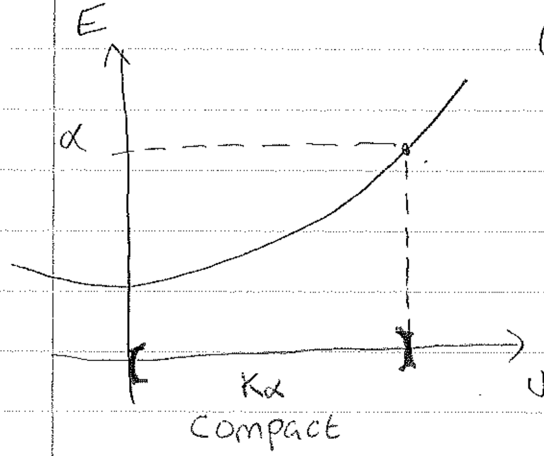
2)

- Basic test for u being a minimizer:
  is $DE(u)=0$?

- u's that satisfy this are critical points

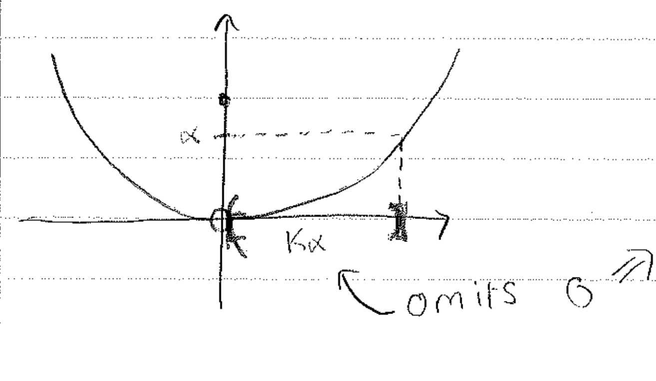
? Question: Do there exist minimizers / critical points?
↳ Are they unique?
↳ Are there solutions to the EL-equation?
($DE(u)=0$ is the Euler-Lagrange equation)

Theorem: (Most general set up)
- M is a topological Hausdorff space
- $E: M \to \mathbb{R} \cup \{\infty\}$ satisfies,
  $\forall \alpha \in \mathbb{R}: K_\alpha = \{u \in M : E(u) \leq \alpha\}$ is compact
  (or sequentially compact)



$\Rightarrow$ E is bounded uniformly from below on M and attains its minimum

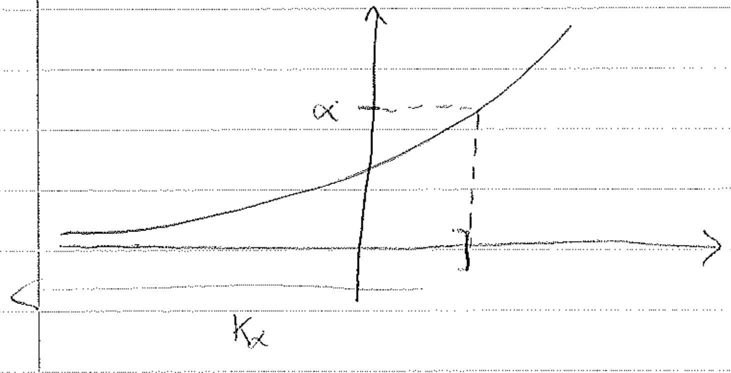
Counter example: 1) Consider $E(x)=x^2$, $x \in [-1,1] \setminus \{0\}$
where $E(0)=1$



half open interval, not compact

3)

2) $E(x) = e^x$, $x \in \mathbb{R}$



K_α is an unbounded interval $\Rightarrow$ not compact

in this example, $\inf E = 0$, no minimizers

Proof of Theorem: Assume $E \not\equiv \infty$ (since we are seeking minimizers)

Let $\alpha_0 = \inf_M E \geq -\infty$ (*)

Let $\{\alpha_m\}_{m \in \mathbb{N}}$ strictly decreasing $\alpha_m \searrow \alpha_0$

$K_m := K_{\alpha_m}$ compact (by our assumption)
↳ also $K_m \supset K_{m+1} \supset K_{m+2} \supset \dots$

By compactness $\bigcap_m K_m \neq \emptyset \Rightarrow u \in \bigcap_m K_m$

i.e.: $E(u) \leq \alpha_m \; \forall m$

as $m \to \infty$: $E(u) \leq \alpha_0$

By (*) $E(u) = \alpha_0$ and u is a minimizer

∎

For PDES our space M is often a subset of a vector space of functions, and instead of assumptions on sublevel sets K_α it is easier to check the following sufficient conditions:

4)

Theorem : • For V a reflexive Banach Space
(i.e : $W^{1,p}(\Omega)$, $p \neq 1, \infty$)
• With $M \subset V$ weakly closed
• $E : M \to \mathbb{R} \cup \{\infty\}$ such that
1) coercivity : $E(u) \to \infty$ as $\|u\|_V \to \infty$

2) $\forall u \in M$ $\forall \{u_m\} \subseteq M$ with
$u_m \rightharpoonup u$ weakly then

lower semi continuity

$E(u) \leq \liminf_{m \to \infty} E(u_m)$

Then E is bounded from below and there exists a minimizer

Examples :

1) $E(u) = \|u\|^p_{W^{1,p}(\Omega)}$ or $\|u\|_{W^{1,p}(\Omega)}$ $\forall p \in (1, \infty]$
more generally,
$E(u) = \|u\|_V$ for V Banach space

2) $E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega f u$, $p \in (1,\infty)$
, $E : W_0^{1,p}(\Omega) \to \mathbb{R}$

$= \underbrace{\frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}}_{\text{this term satisfies 2) in the above thm}} - \underbrace{\int_\Omega f u}$

If $f \in W_0^{1,p}(\Omega)^* = W^{-1,q}(\Omega)$
for $\frac{1}{p} + \frac{1}{q} = 1$
(f is a linear functional)
then this term also satisfies 2)

Indeed, $\langle f, u\rangle = \int_\Omega f u$
$= \lim_{m \to \infty} \langle f, u_m\rangle$ if $u_m \rightharpoonup u$

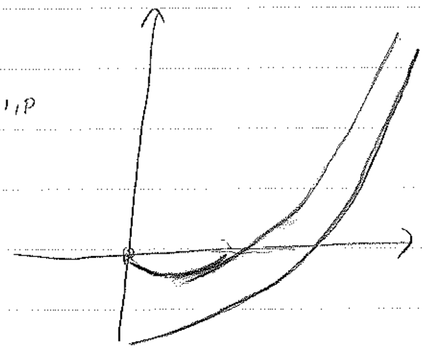
5)

Thus $E(u)$ Satisfies condition 2).

For condition 1) we need to show
$E(u) \to \infty$ as $\|u\|_{W^{1,p}(\Omega)} \to \infty$

$\hookrightarrow$ Idea: $x^p - cx \to \infty$ as $x \to \infty$ if $p > 1$

$$E(u) = \frac{1}{p}\|u\|_{W^{1,p}}^p - \int_\Omega fu$$



$$\geq \frac{1}{p}\|u\|_{W^{1,p}}^p - \underbrace{\|f\|_{W^{1,p*}}}_{= C < \infty}\|u\|_{W^{1,p}}$$

$$= \frac{1}{p}\|u\|_{W^{1,p}}^p - C\|u\|_{W^{1,p}} \quad (-)$$

$$\geq \alpha\|u\|_{W^{1,p}}^p - C \quad (-) \text{ for some } \alpha, C > 0$$

$$\to \infty \quad \text{as } \|u\| \to \infty$$

Thus 1) and 2) are satisfied and there exists a minimizer $u \in W_0^{1,p}(\Omega)$

$\square$

What does $DE(u) = 0$, the Euler Lagrange equation look like? (for $p \geq 2$)

$$\langle DE(u), v \rangle := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u + \varepsilon v)$$

$\forall\, v \in W_0^{1,p}(\Omega)$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{p}\int_\Omega (\nabla(u+\varepsilon v))^p dx - \int_\Omega f(u+\varepsilon v)\, dx$$

6)

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left\{ \underbrace{\frac{1}{p}\int_\Omega (\nabla u + \varepsilon \nabla v)^p \, dx} - \underbrace{\cancel{\int_\Omega f u \, dx}}_{\text{no } \varepsilon} - \varepsilon \int_\Omega f v \, dx \right\}$$

$$= \frac{1}{p}\int_\Omega |\nabla u|^p \left(\frac{\nabla u}{|\nabla u|} + \varepsilon \frac{\nabla v}{|\nabla u|} \right)^p$$

Taylor expand
diff + set $\varepsilon = 0$

$$= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx - \int_\Omega f v \, dx$$

Now, if u is a minimizer $\Rightarrow \langle DE(u), v \rangle = 0$ $\forall v \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v - \int_\Omega f v = 0$$

(This is the weak formulation of

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega \;)$$

Indeed if u is sufficiently smooth, such that $|\nabla u|^{p-2} \nabla u \in W^{1,2}(\Omega)$ then we can integrate by parts:

$$-\int_\Omega \nabla \cdot (|\nabla u|^{p-2} \nabla u) v + \underbrace{\cancel{\int_{\partial\Omega} \cdots}}_{0 \text{ since } v = 0 \text{ on } \partial\Omega} - \int_\Omega f v = 0$$

i.e. $-\int_\Omega \left\{ \nabla \cdot (|\nabla u|^{p-2} \nabla u) + f \right\} v \, dx = 0 \quad \forall v \in W^{1,p}$

$$\Longleftrightarrow \nabla \cdot (|\nabla u|^{p-2} \nabla u) + f = 0 \quad \text{in } (W_0^{1,p})^*$$

7)

Conclusion: $\nabla \cdot (|\nabla u|^{p-2} \nabla u) + f = 0$

admits a $\underset{\text{weak}}{\wedge}$ solution $u \in W_0^{1,p}(\Omega)$

One can easily show that it is unique

↳ If $u_1, u_2 \in W_0^{1,p}(\Omega)$ are two solutions then,

$$\int_\Omega \left(|\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla v - |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla v \right) \cancel{- f v + f v} = 0$$

Choose $v = u_1 - u_2$

$$\underbrace{\int_\Omega \left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) \cdot \nabla (u_1 - u_2)}_{\geq\!\!/} = 0$$

$$0 \leq \int_\Omega |\nabla (u_1 - u_2)|^p \, dx$$

$\Rightarrow u_1 = u_2$

$\Rightarrow$ u is unique !!

Another example: $\Omega \subseteq \mathbb{R}^n$ bounded domain and $S \subseteq \mathbb{R}^n$ compact

Consider functions $u : \Omega \to \mathbb{R}^n$ such that $u(x) \in S$ a.e

Define, $W^{1,2}(\Omega, S) = \{ u \in W^{1,2}(\Omega) : u(\Omega) \subseteq S \text{ a.e} \}$

↑ takes values in $\mathbb{R}^N$

Consider the energy,

$$E(u) = \int_\Omega g_{ij}(u) \nabla u^i \cdot \nabla u^j \, dx$$

8)

This energy functional appears in the study of minimal hyper surfaces in a riemann manifold: $E(u) \to \min$

The relevent M to apply our theorem is

$$M = \{ u \in W^{1,2}(\Omega, S) : u = u_0 \text{ on } \partial\Omega \}$$