Morse-Tompkins [1] and Shiffman [1] by a less direct, topological reasoning. However, since this material has been covered very extensively elsewhere (see Struwe [18]), here we will confine ourselves to the above remarks.

For an introduction to Plateau's problem and minimal surfaces, see for instance, Osserman [1], or consult the encyclopedic book by Nitsche [2].

A truely remarkable – popular and profound – book on the subject is available by Hildebrandt-Tromba [1].

3. Compensated Compactness

As noted in Remark 1.8, it is conceivable that in the vector-valued case lower semi-continuity results may hold true under a weaker convexity assumption than in Theorem 1.6, provided suitable structure conditions are satisfied by the functional in variation. Weakening the convexity hypothesis is necessary, for instance, in dealing with problems arising in 3-dimensional elasticity, where we encounter energy functionals $\int_\Omega W(\nabla u)\,dx$ with a stored energy function W depending on the determinant, the minors and the eigenvalues of the deformation gradient ∇u. Since infinite volume distortion for elastic materials will afford an infinite amount of energy, it is natural to suppose that $W \to \infty$ if either $\det(\nabla u) \to 0$ or $\det(\nabla u) \to \infty$; hence W cannot be convex in ∇u. However, there is a large class of materials that can be described by *polyconvex* stored energy functions, which are of the form

$$W(\nabla u) = f(\text{subdeterminants of } \nabla u),$$

where f is convex in each of its variables. John Ball [1] was the first to see that lower semi-continuity results will hold for such functionals. The difficulty, of course, lies in proving, for instance, weak convergence $\det(\nabla u_m) \rightharpoonup \det(\nabla u)$ for a sequence $u_m \rightharpoonup u$ weakly in $H^{1,3}(\Omega, \mathbb{R}^3)$. Questions of this type had been investigated by Reshetnyak [1], [2]. A general frame for studying such problems is provided by the compensated compactness scheme of Murat and Tartar.

The basic principle of the compensated compactness method is given in the following lemma; see Tartar [2; p. 270 f.].

3.1 The compensated compactness lemma. *Let Ω be a domain in $\mathbb{R}^n$ and suppose that*
(1°) $u_m = \left(u_m^1, \dots, u_m^N\right) \rightharpoonup u$ weakly in $L^2(\Omega; \mathbb{R}^N)$.
(2°) The set $\left\{\sum_{j,k} a_{jk} \frac{\partial u_m^j}{\partial x_k}\ ;\ m \in \mathbb{N}\right\}$ is relatively compact in $H^{-1}_{loc}(\Omega; \mathbb{R}^L)$ for a set of vectors $a_{jk} \in \mathbb{R}^L;\ 1 \le j \le N,\ 1 \le k \le n$. Let

$$\Lambda = \left\{\lambda \in \mathbb{R}^N\ ;\ \sum_{j,k} a_{jk}\lambda_j\xi_k = 0 \textit{ for some } \xi \in \mathbb{R}^n \setminus \{0\}\right\}$$

and let Q be a (real) quadratic form such that $Q(\lambda) \geq 0$ for all $\lambda \in \Lambda$. Regarding $Q(u_m) \in L^1(\Omega)$ as Radon measures $Q(u_m)dx \in \big(C^0(\Omega')\big)^*$, we may assume that $\big(Q(u_m)\big)$ converges weak, locally.*
Then on any $\Omega' \subset\subset \Omega$ we have

$$\text{weak}^* - \lim_{m\to\infty} Q\big(u_m\big) \geq Q(u)$$

in the sense of measures. In particular, if $Q(\lambda) = 0$ for all $\lambda \in \Lambda$, then

$$\text{weak}^* - \lim_{m\to\infty} Q\big(u_m\big) = Q(u)$$

locally, in the sense of measures.

Proof. Choose $\varphi \in C^0(\Omega)$ with compact support and such that $0 \leq \varphi \leq 1$. We must show that

$$\liminf_{m\to\infty} \int_\Omega Q\big(u_m\big)\varphi^2 \, dx \geq \int_\Omega Q(u)\varphi^2 \, dx \ .$$

By translation we may assume that $u = 0$. Moreover, upon replacing u_m by $u_m\varphi$ we may assume that the supports of u_m lie in a fixed cube $K \subset\subset \Omega$ and that $\varphi \equiv 1$ on K. By translation and scaling, moreover, we can obtain $K = [0, 2\pi]^n$. Let

$$u_m(x) = \sum_{\alpha\in\mathbb{Z}^n} \mu_{m,\alpha} e^{i\alpha\cdot x} \ , \qquad \mu_{m,\alpha} = \big(\mu^1_{m,\alpha}, \ldots, \mu^N_{m,\alpha}\big) \in \mathbb{C}^N \ ,$$

be the Fourier expansion for u_m. Since u_m is real, we have $\mu_{m,\alpha} = \overline{\mu}_{m,-\alpha}$. The assertion then is equivalent to showing that

$$\liminf_{m\to\infty} \frac{1}{(2\pi)^n} \int_K Q\big(u_m\big) \, dx = \liminf_{m\to\infty} \sum_{\alpha\in\mathbb{Z}^n} \Big(Q\big(\mathrm{Re}\ \mu_{m,\alpha}\big) + Q\big(\mathrm{Im}\ \mu_{m,\alpha}\big)\Big) \geq 0 \ .$$

By weak convergence $u_m \rightharpoonup 0$ in L^2 we infer that $\sum_{\alpha\in\mathbb{Z}^n} |\mu_{m,\alpha}|^2 \leq c < \infty$ and $\mu_{m,\alpha} \to 0$ as $m \to \infty$, uniformly on bounded sets of indices α.

Moreover, by (2°) the set

$$\left\{ \sum_{\alpha\in\mathbb{Z}^n} \sum_{j,k} a_{jk}\mu^j_{m,\alpha} \, \alpha_k \, e^{i\,\alpha\cdot x} \ ; \ m \in \mathbb{N} \right\}$$

is relatively compact in H^{-1}, which implies that

$$\sum_{\substack{\alpha\in\mathbb{Z}^n \\ |\alpha|\geq\alpha_0}} \frac{\Big|\sum_{j,k} a_{jk}\mu^j_{m,\alpha} \, \alpha_k\Big|^2}{1 + |\alpha|^2} \to 0,$$

as $\alpha_0 \to \infty$, uniformly in $m \in \mathbb{N}$. But this means that $\mu_{m,\alpha}$ can be decomposed $\mu_{m,\alpha} = \lambda_{m,\alpha} + \nu_{m,\alpha}$ with Re $\lambda_{m,\alpha}$, Im $\lambda_{m,\alpha} \in \Lambda$ and $\sum_{|\alpha|\geq\alpha_0} |\nu_{m,\alpha}|^2 \to 0$ as $\alpha_o \to \infty$, uniformly in $m \in \mathbb{N}$.

Indeed, for any $\alpha \in \mathbb{Z}^n$, $m \in \mathbb{N}$ decompose $\mu_{m,\alpha} = \lambda_{m,\alpha} + \nu_{m,\alpha}$, where Re $\lambda_{m,\alpha}$, Im $\lambda_{m,\alpha} \in \Lambda$ and $|\nu_{m,\alpha}|^2$ is minimal among all decompositions of this kind. We claim that for any $\varepsilon > 0$ there exist constants $C = C(\varepsilon)$, $\alpha_0 = \alpha_0(\varepsilon)$ such that for $|\alpha| \geq \alpha_0$ we can bound

$$(3.1) \qquad |\nu_{m,\alpha}|^2 \leq C \left| \sum_{j,k} a_{jk} \mu^j_{m,\alpha} \frac{\alpha_k}{\sqrt{1+|\alpha|^2}} \right|^2 + \varepsilon |\mu_{m,\alpha}|^2$$

uniformly in $m \in \mathbb{N}$.

Otherwise there exists $\varepsilon > 0$ and a sequence $\alpha = \alpha(l)$, $l \in \mathbb{N}$, with $|\alpha(l)| \geq l$, and $m = m(l)$ such that for all l there holds

$$(3.2) \qquad |\mu_{m,\alpha}|^2 \geq |\nu_{m,\alpha}|^2 \geq l \left| \sum_{j,k} a_{jk} \mu^j_{m,\alpha} \frac{\alpha_k}{\sqrt{1+|\alpha|^2}} \right|^2 + \varepsilon \, |\mu_{m,\alpha}|^2 \ .$$

(The first inequality follows from the choice of $\nu_{m,\alpha}$ above.) Let $\xi(l)$, $\eta(l)$ be the unit vectors

$$\xi(l) = \frac{\alpha(l)}{\sqrt{1+|\alpha(l)|^2}} \in S^{n-1}, \ \eta(l) = \frac{\mu_{m(l),\alpha(l)}}{|\mu_{m(l),\alpha(l)}|} \in S^{N-1} \ ,$$

and denote by $A(l) \colon \mathbb{R}^N \to \mathbb{R}^L$ the linear map

$$\eta \mapsto \sum_{j,k} a_{jk} \eta_j \xi_k(l) \ .$$

We may assume that $\xi(l) \to \xi$ and $A(l) \to A$ as $l \to \infty$. Likewise, we may suppose that $\eta(l) \to \eta$. Passing to the limit in (3.2) it follows that $\eta \in \mathit{ker}\ A$; that is, $\eta \in \Lambda$. Projecting $\mu_{m,\alpha}$ onto $\mathit{ker}\ A$ for all $\alpha = \alpha(l)$, $m = m(l)$ we hence obtain a decomposition $\mu_{m,\alpha} = \tilde{\lambda}_{m,\alpha} + \tilde{\nu}_{m,\alpha}$ with Re $\tilde{\lambda}_{m,\alpha}$, Im $\tilde{\lambda}_{m,\alpha} \in \mathit{ker}\ A \subset \Lambda$ and

$$\begin{aligned} |\tilde{\nu}_{m,\alpha}|^2 &\leq C |A\mu_{m,\alpha}|^2 \leq C |A(l)\mu_{m,\alpha}|^2 + o(1) |\mu_{m,\alpha}|^2 \\ &= C \left| \sum_{j,k} a_{jk} \mu^j_{m,\alpha} \frac{\alpha_k}{\sqrt{1+|\alpha|^2}} \right|^2 + o(1) |\mu_{m,\alpha}|^2 \ , \end{aligned}$$

where $o(1) \to 0$ $(l \to \infty)$. But by defintion $|\nu_{m,\alpha}|^2 \leq |\tilde{\nu}_{m,\alpha}|^2$, and we obtain the desired contradiction to assumption (3.2). Hence for any $\varepsilon > 0$, any $\alpha_0 \geq \alpha_0(\varepsilon)$:

$$\begin{aligned}
&\liminf_{m\to\infty} \sum_{\alpha\in\mathbb{Z}^n} \Big(Q\big(\mathrm{Re}\ \mu_{m,\alpha}\big) + Q\big(\mathrm{Im}\ \mu_{m,\alpha}\big)\Big) \\
&\qquad = \liminf_{m\to\infty} \sum_{|\alpha|\geq\alpha_0} \Big(Q\big(\mathrm{Re}\ \mu_{m,\alpha}\big) + Q\big(\mathrm{Im}\ \mu_{m,\alpha}\big)\Big) \\
&\qquad \geq \liminf_{m\to\infty} \sum_{|\alpha|\geq\alpha_0} \Big(Q\big(\mathrm{Re}\ \lambda_{m,\alpha}\big) + Q\big(\mathrm{Im}\ \lambda_{m,\alpha}\big) - c\varepsilon\big|\mu_{m,\alpha}\big|^2\Big) + o(1) \\
&\qquad \geq o(1) - c\varepsilon\ ,
\end{aligned}$$

where $o(1) \to 0$ as $\alpha_0 \to \infty$. Thus, the assertion of the lemma follows as we first let $\alpha_0 \to \infty$ and then $\varepsilon \to 0$. □

As an application we mention the following well-known result.

3.2 The Div-Curl Lemma. *Suppose $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ weakly in $L^2(\Omega;\mathbb{R}^3)$ on a domain $\Omega \subset \mathbb{R}^3$ while the sequences* (div u_m) *and* (curl v_m) *are relatively compact in $H^{-1}(\Omega)$. Then for any $\varphi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega u_m \cdot v_m \varphi\, dx \to \int_\Omega u \cdot v\varphi\, dx$$

as $m \to \infty$.

Proof. Let $w_m = (u_m, v_m) \in L^2(\Omega;\mathbb{R}^6)$, and determine coefficients $a_{jk} \in \mathbb{R}^4$ such that $\sum_{jk} a_{jk}\frac{\partial w_m^j}{\partial x_k} = (\mathrm{div}\ \ u_m,\ \ \mathrm{curl}\ \ v_m)$. Let Q be the quadratic form $Q(u,v) = u\cdot v$, acting on vectors $w = (u,v) \in \mathbb{R}^6$. Note $a_{jk} = \big(\delta_{jk}, (\varepsilon_{ijk})_{1\leq i\leq 3}\big)$ where $\delta_{jk} = 1$ if $j = k$, and $\delta_{jk} = 0$ else, $\varepsilon_{123} = 1$ and $\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki}$. Hence

$$\begin{aligned}
\Lambda &= \big\{\lambda = (\mu,\nu) \in \mathbb{R}^6\ ;\ \exists\xi \in \mathbb{R}^3 \setminus \{0\} :\ (\xi\cdot\mu\ ,\ \xi\wedge\nu) = 0\big\} \\
&= \big\{\lambda = (\mu,\nu) \in \mathbb{R}^6\ ;\ \mu\cdot\nu = 0\big\}\ ,
\end{aligned}$$

and $Q \equiv 0$ on Λ. Thus the assertion follows from Lemma 3.1. □

The div-curl Lemma 3.2 shows how additional bounds on *some* derivatives allow one to prove continuity of nonlinear expressions (bi-linear in the above example) under weak convergence.

Phrased somewhat differently, the reason for the convergence result stated in the div-curl lemma to hold is an implicit divergence structure. This stucture can be brought out more clearly in the language of differential forms. For simplicity, we assume that all functions involved are periodic of period 1 in each variable. In this case, we may regard $\Omega = [0,1]^n$ as a fundamental domain for the flat n-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Let d, d^* be the exterior differential and co-differential, respectively. We consider 1-forms $u_m \rightharpoonup u$ in L^2, $v_m \rightharpoonup v$

in L^2 weakly as $m \to \infty$ with $(d^* u_m), (dv_m)$ relatively compact in H^{-1}. We may assume $u = 0, v = 0$. By Hodge decomposition we have

$$\begin{aligned} u_m &= da_m + d^* b_m + c_m, \\ v_m &= df_m + d^* g_m + h_m, \end{aligned}$$

where c_m, h_m are harmonic 1-forms, $db_m = dg_m = 0$, and with $a_m \rightharpoonup 0, b_m \rightharpoonup 0, f_m \rightharpoonup 0, g_m \rightharpoonup 0$ weakly in $H^{1,2}(T^n)$, $c_m \rightharpoonup 0$, $h_m \rightharpoonup 0$ weakly in $L^2(T^n)$. Since the space of harmonic 1-forms on T^n is compact, in fact, $c_m \to 0$ and $h_m \to 0$ smoothly as $m \to \infty$. Moreover, since the sequences

$$\Delta a_m = d^* u_m, \Delta g_m = dv_m$$

are relatively compact in H^{-1}, it follows that $(da_m), (d^* g_m)$ are precompact in L^2, and we conclude that

$$u_m = d^* b_m + o(1), v_m = df_m + o(1),$$

where $o(1) \to 0$ in L^2 as $m \to \infty$. Moreover, using the Hodge $*$-operator and denoting by "." contraction of forms, we have

$$u_m \cdot v_m \, dx + o(1) = *(d^* b_m \cdot df_m) = (d * b_m) \wedge df_m = d\big((*b_m) \wedge df_m\big) \ ,$$

thus exhibiting the asserted divergence structure. Since by Rellich's compactness result $b_m \to 0$ strongly in L^2 as $m \to \infty$, it is now trivial to pass to the limit in the expression

$$\begin{aligned} \int_{T^n} u_m \cdot v_m \varphi \, dx &= \int_{T^n} d\big((*b_m \wedge df_m\big)\varphi + o(1) \\ &= (-1)^n \int_{T^n} (*b_m) \wedge df_m \wedge d\varphi + o(1) = o(1), \end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$.

A divergence structure is also the crucial ingredient in the applications that follow.

Applications in Elasticity

The most important applications of the compensated compactness method so far are in elasticity and hyperbolic systems, see Ball [1], [2], DiPerna [1]. DiPerna-Majda have applied compensated compactness methods to obtain the existence of weak solutions to the Euler equations for incompressible fluids, see for instance DiPerna-Majda [1]. Our interest lies with the extensions of the direct methods that compensated compactness implies. Thus we will concentrate on Ball's lower semi-continuity results for polyconvex materials in elasticity.

3.3 Theorem. *Suppose W is a function on (3×3)-matrices Φ, given by*

$$W(\Phi) = g(\Phi, adj\ \Phi, \det\ \Phi)$$

where g is a convex non-negative function in the sub-determinants of Φ. Let Ω be a domain in $\mathbb{R}^3$ and let $u_m,\ u \in H^{1,3}_{loc}(\Omega; \mathbb{R}^3)$. Suppose that $u_m \rightharpoondown u$ weakly in $H^{1,3}(\Omega'; \mathbb{R}^3)$ while $\det(\nabla u_m) \rightharpoondown h$ weakly in $L^1(\Omega')$ for all $\Omega' \subset\subset \Omega$, where $h \in L^1_{loc}(\Omega)$. Then

$$\int_\Omega W(\nabla u)\,dx \le \liminf_{m\to\infty} \int_\Omega W(\nabla u_m)\,dx\ .$$

Proof. The proof of Theorem 1.6 can be carried over once we show that under the hypotheses made

$$\begin{aligned} adj(\nabla u_m) &\rightharpoondown adj(\nabla u) \\ \det(\nabla u_m) &\rightharpoondown \det(\nabla u) \end{aligned}$$

weakly in $L^1(\Omega')$ for all $\Omega' \subset\subset \Omega$. The first assertion is a consequence of the divergence structure of the adjoint matrix $A_m = adj\,(\nabla u_m)$. Indeed, if indices i, j are counted *modulo* 3 we have

$$\begin{aligned} A^{ij}_m &= \frac{\partial u^{i+1}_m}{\partial x^{j+1}} \frac{\partial u^{i+2}_m}{\partial x^{j+2}} - \frac{\partial u^{i+2}_m}{\partial x^{j+1}} \frac{\partial u^{i+1}_m}{\partial x^{j+2}} \\ &= \frac{\partial}{\partial x^{j+1}} \left(u^{i+1}_m \frac{\partial u^{i+2}_m}{\partial x^{j+2}} \right) - \frac{\partial}{\partial x^{j+2}} \left(u^{i+1}_m \frac{\partial u^{i+2}_m}{\partial x^{j+1}} \right) . \end{aligned}$$

Fix $\Omega' \subset\subset \Omega$. Note that $\left(A^{ij}_m\right)$ is bounded in $L^{3/2}(\Omega')$. Hence we may assume that $A^{ij}_m \rightharpoondown A^{ij}$ weakly in $L^{3/2}(\Omega')$. Moreover, by Rellich's theorem $u_m \to u$ in $L^3(\Omega')$, whence $u_m \nabla u_m \rightharpoondown u \nabla u$ weakly in $L^{3/2}(\Omega')$. By continuity of the distributional derivative with respect to weak convergence therefore $A^{ij}_m \rightharpoondown \left(adj(\nabla u)\right)^{ij}$ in the sense of distributions. Finally, by uniqueness of the distributional limit, $A^{ij} = \left(adj(\nabla u)\right)^{ij}$, and $adj(\nabla u_m) \rightharpoondown adj(\nabla u)$ weakly in $L^{3/2}(\Omega')$, in particular weakly in $L^1(\Omega')$, as claimed.

Similarly, expanding the determinant along the first row, we have

$$\begin{aligned} \det(\nabla u_m) = &\frac{\partial u^1_m}{\partial x^1} \left[\frac{\partial u^2_m}{\partial x^2} \frac{\partial u^3_m}{\partial x^3} - \frac{\partial u^2_m}{\partial x^3} \frac{\partial u^3_m}{\partial x^2} \right] \\ &- \frac{\partial u^1_m}{\partial x^2} \left[\frac{\partial u^2_m}{\partial x^1} \frac{\partial u^3_m}{\partial x^3} - \frac{\partial u^2_m}{\partial x^3} \frac{\partial u^3_m}{\partial x^1} \right] \\ &+ \frac{\partial u^1_m}{\partial x^3} \left[\frac{\partial u^2_m}{\partial x^1} \frac{\partial u^3_m}{\partial x^2} - \frac{\partial u^2_m}{\partial x^2} \frac{\partial u^3_m}{\partial x^1} \right] . \end{aligned}$$

Again, a divergence structure emerges, if we rewrite this as

$$\begin{aligned}\det(\nabla u_m) = &\frac{\partial}{\partial x^1}\left(u_m^1\left[\frac{\partial u_m^2}{\partial x^2}\frac{\partial u_m^3}{\partial x^3} - \frac{\partial u_m^2}{\partial x^3}\frac{\partial u_m^3}{\partial x^2}\right]\right)\\ &- \frac{\partial}{\partial x^2}\left(u_m^1\left[\frac{\partial u_m^2}{\partial x^1}\frac{\partial u_m^3}{\partial x^3} - \frac{\partial u_m^2}{\partial x^3}\frac{\partial u_m^3}{\partial x^1}\right]\right)\\ &+ \frac{\partial}{\partial x^3}\left(u_m^1\left[\frac{\partial u_m^2}{\partial x^1}\frac{\partial u_m^3}{\partial x^2} - \frac{\partial u_m^2}{\partial x^2}\frac{\partial u_m^3}{\partial x^1}\right]\right) .\end{aligned}$$

Thus convergence $\det(\nabla u_m) \to \det(\nabla u)$ in the sense of distributions follows by weak convergence in $L^{3/2}(\Omega')$ of the terms in brackets [- -], proved above, strong convergence $u_m \to u$ in $L^3(\Omega')$, and weak continuity of the distributional derivative. Finally, by uniqueness of the weak limit in the distribution sense, it follows that $\det(\nabla u) = h$ and $\det(\nabla u_m) \rightharpoondown \det(\nabla u)$ weakly in L^1_{loc}, as claimed. $\square$

Observe that in the language of differential forms the divergence structure of a Jacobian or its minors is even more apparent. In fact, for any smooth function $u = (u^1, u^2, u^3)\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ we have

$$\det(\nabla u)\, dx = *\det(\nabla u) = du^1 \wedge du^2 \wedge du^3 ,$$

where d denotes exterior derivative and where $*$ denotes the Hodge star operator (which in this case converts a function on Ω into a 3-form). Now $dd = 0$, and therefore

$$du^1 \wedge du^2 \wedge du^3 = d(u^1 du^2 \wedge du^3) ,$$

which immediately implies the asserted divergence structure. Moreover, this result (and therefore Theorem 3.3) generalizes to any dimension n, for $u_m \rightharpoondown u$ weakly in $H^{1,n}_{loc}(\Omega;\mathbb{R}^n)$ with $\det(\nabla u_m) \rightharpoondown h$ weakly in $L^1_{loc}(\Omega)$ as $m \to \infty$.

The assumption $\det(\nabla u_m) \rightharpoondown h \in L^1_{loc}(\Omega)$ at first sight may appear rather awkward. However, examples by Ball-Murat [1] show that weak $H^{1,3}$-convergence in general does not imply weak L^1-convergence of the Jacobian. This difficulty does not arise if we assume weak convergence in $H^{1,3+\varepsilon}$ for some $\varepsilon > 0$.

Hence, adding appropriate growth conditions on W to ensure coerciveness of the functional $\int_\Omega W(\nabla u)\, dx$ on the space $H^{1,3+\varepsilon}(\Omega;\mathbb{R}^3)$ for some $\varepsilon > 0$, from Theorem 3.3 the reader can derive existence theorems for deformations of elastic materials involving polyconvex stored energy functions. As a further reference for such results, see Ciarlet [1] or Dacorogna [1], [2]. Recently, more general results on weak continuity of determinants and corresponding existence theorems in nonlinear elasticity have been obtained by Giaquinta-Modica-Souček [1] and Müller [1], [2], [3].

The regularity theory for problems in nonlinear elasticity is still evolving. Some material can be found in the references cited above. In particular, the question of cavitation of elastic materials has been studied. See for instance Giaquinta-Modica-Souček [1].

Convergence Results for Nonlinear Elliptic Equations

We close this section with another simple and useful example of how compensated compactness methods may be applied in a nonlinear situation. The following result is essentially "Murat's lemma" from Tartar [2, p. 278]:

3.4 Theorem. *Suppose $u_m \in H_0^{1,2}(\Omega)$ is a sequence of solutions to an elliptic equation*

$$\begin{aligned} -\Delta u_m &= f_m \qquad \text{in } \Omega \\ u_m &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

in a smooth and bounded domain Ω in $\mathbb{R}^n$. Suppose $u_m \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega)$ while (f_m) is bounded in $L^1(\Omega)$. Then for a subsequence $m \to \infty$ we have $\nabla u_m \to \nabla u$ in $L^q(\Omega)$ for any $q < 2$, and $\nabla u_m \to \nabla u$ pointwise almost everywhere.

Proof. Choose $p > n$ and let $\varphi_m \in H_0^{1,p}(\Omega)$ satisfy

$$\|\varphi_m\|_{H_0^{1,p}} \leq 1$$

$$\int_\Omega (\nabla u_m - \nabla u)\nabla\varphi_m \, dx = \sup_{\varphi \in H_0^{1,p}(\Omega),\ \|\varphi\|_{H_0^{1,p}} \leq 1} \int_\Omega (\nabla u_m - \nabla u)\nabla\varphi \, dx \ .$$

By the Calderón-Zygmund inequality in L^p, see Simader[1], the latter

$$\sup_{\varphi \in H_0^{1,p}(\Omega),\ \|\varphi\|_{H_0^{1,p}} \leq 1} \int_\Omega (\nabla u_m - \nabla u)\nabla\varphi \, dx \geq c^{-1}\|u_m - u\|_{H_0^{1,q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

On the other hand, by Sobolev's embedding $H_0^{1,p}(\Omega) \hookrightarrow C^{1-\frac{n}{p}}(\overline{\Omega})$. Hence by the Arzéla-Ascoli theorem we may assume that $\varphi_m \rightharpoonup \varphi$ weakly in $H_0^{1,p}(\Omega)$ and uniformly in $\overline{\Omega}$. (See Theorem A.5.) Thus

$$\begin{aligned} \int_\Omega (\nabla u_m - \nabla u)\nabla\varphi_m \, dx &= \int_\Omega (\nabla u_m - \nabla u)(\nabla\varphi_m - \nabla\varphi) \, dx + o(1) \\ &= \lim_{l\to\infty} \int_\Omega (\nabla u_m - \nabla u_l)(\nabla\varphi_m - \nabla\varphi) \, dx + o(1) \\ &= \lim_{l\to\infty} \int_\Omega (f_m - f_l)(\varphi_m - \varphi) \, dx + o(1) \\ &\leq 2\sup_{l\in\mathbb{N}} \|f_l\|_{L^1}\|\varphi_m - \varphi\|_{L^\infty} + o(1) = o(1), \end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$.

It follows that $\nabla u_m \to \nabla u$ in $L^{q_0}(\Omega)$ for some $q_0 \geq 1$. But then, by Hölder's inequality, for any $q < 2$ we have

$$\|\nabla u_m - \nabla u\|_{L^q} \leq \|\nabla u_m - \nabla u\|_{L^1}^{\gamma} \|\nabla u_m - \nabla u\|_{L^2}^{1-\gamma} \to 0\,,$$

where $\frac{1}{q} = \gamma + \frac{1-\gamma}{2}$. □

Results like Theorem 3.4 are needed if one wants to solve nonlinear partial differential equations

$$-\Delta u = f(x, u, \nabla u) \tag{3.3}$$

with quadratic growth

$$|f(x, u, p)| \leq c(1 + |p|^2)$$

by approximation methods. Assuming some uniform control on approximate solutions u_m of (3.3) in $H^{1,2}$, Theorem 3.4 assures that

$$f(x, u_m, \nabla u_m) \to f(x, u, \nabla u) \ \text{ almost everywhere,}$$

where u is the weak limit of a suitable sequence (u_m). Given some further structure conditions on f, then there are various ways of passing to the limit $m \to \infty$ in Equation (3.3); see, for instance, Frehse [2] or Evans [2].

3.5 Example. As a model problem consider the equation

$$A(u) = -\Delta u + u|\nabla u|^2 = h \quad \text{in } \Omega \tag{3.4}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{3.5}$$

on a smooth and bounded domain $\Omega \subset \mathbb{R}^n$, with $h \in L^\infty(\Omega)$. This is a special case of a problem studied by Bensoussan-Boccardo-Murat [1; Theorem 1.1, p. 350]. Note that the nonlinear term $g(u, p) = u|p|^2$ satisfies the condition

$$g(u, p)u \geq 0\ . \tag{3.6}$$

Approximate g by functions

$$g_\varepsilon(u, p) = \frac{g(u, p)}{1 + \varepsilon|g(u, p)|}\ , \varepsilon > 0\ ,$$

satisfying $|g_\varepsilon| \leq \frac{1}{\varepsilon}$ and $g_\varepsilon(u, p) \cdot u \geq 0$ for all u, p.

Now, since g_ε is uniformly bounded, the map $H_0^{1,2}(\Omega) \ni u \mapsto g_\varepsilon(u, \nabla u) \in H^{-1}(\Omega)$ is compact and bounded for any $\varepsilon > 0$. Denote $A_\varepsilon(u) = -\Delta u + g_\varepsilon(u, \nabla u)$ the perturbed operator A. By Schauder's fixed point theorem, see, for instance, Deimling [1; Theorem 8.8, p. 60], applied to the map $u \mapsto (-\Delta)^{-1}\big(h - g_\varepsilon(u, \nabla u)\big)$ on a sufficiently large ball in $H_0^{1,2}(\Omega)$, there is a solution $u_\varepsilon \in H_0^{1,2}(\Omega)$ of the equation $A_\varepsilon u_\varepsilon = h$ for any $\varepsilon > 0$. In addition, since $g_\varepsilon(u, p) \cdot u \geq 0$ we have

$$\|u_\varepsilon\|^2_{H_0^{1,2}} \le \langle u_\varepsilon, A_\varepsilon u_\varepsilon\rangle = \langle u_\varepsilon, h\rangle \le \|u_\varepsilon\|_{H_0^{1,2}} \|h\|_{H^{-1}} \ ,$$

and (u_ε) is uniformly bounded in $H_0^{1,2}(\Omega)$ for $\varepsilon > 0$. Moreover, since the nonlinear term g_ε satisfies

$$\begin{aligned} g_\varepsilon(u,p) &= \frac{u|p|^2}{1+\varepsilon|u|\,|p|^2} \le \frac{\big(1+|u|^2\big)|p|^2}{1+\varepsilon|u|\,|p|^2} \\ &\le |p|^2 + g_\varepsilon(u,p)u \ , \end{aligned}$$

we also deduce the uniform L^1-bound

$$\begin{aligned} \|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\|_{L^1} &\le \|u_\varepsilon\|^2_{H_0^{1,2}} + \int_\Omega u_\varepsilon g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\, dx \\ &= \langle u_\varepsilon, A_\varepsilon u_\varepsilon\rangle \le c \ . \end{aligned}$$

We may assume that a sequence (u_{ε_m}) as $\varepsilon_m \to 0$ weakly converges in $H_0^{1,2}(\Omega)$ to a limit $u \in H_0^{1,2}(\Omega)$. By Theorem 3.4, moreover, we may assume that $u_m = u_{\varepsilon_m}$ converges strongly in $H_0^{1,q}(\Omega)$ and that u_m and ∇u_m converge pointwise almost everywhere. To show that u weakly solves (3.4), (3.5) we now use the “Fatou lemma technique” of Frehse [2]. As a preliminary step we establish a uniform L^∞-bound for the sequence (u_m).
Multiply the approximate equations by u_m to obtain the differential inequality

$$\begin{aligned} -\Delta\Big(\frac{|u_m|^2}{2}\Big) &\le -\Delta\Big(\frac{|u_m|^2}{2}\Big) + |\nabla u_m|^2 + u_m g_{\varepsilon_m}(u_m, \nabla u_m) \\ &= h u_m \le C(\delta) + \delta \frac{|u_m|^2}{2} \ , \end{aligned}$$

for any $\delta > 0$. Choosing $\delta < \lambda_1$, the first eigenvalue of $-\Delta$ on $H_0^{1,2}(\Omega)$, the weak maximum principle implies a uniform bound for u_m in L^∞. (See Theorem B.7 and its application in Appendix B.)

Next, testing the approximate equations $A_{\varepsilon_m}(u_m) = h$ with $\varphi = \xi \exp(\gamma u_m)$, where $\xi \in C_0^\infty(\Omega)$ is non-negative, upon integrating by parts we obtain

$$\begin{aligned} &\int_\Omega \Big(\gamma|\nabla u_m|^2 + g_{\varepsilon_m}(u_m, \nabla_{u_m})\Big)\xi \exp(\gamma u_m)\, dx \\ &\qquad + \int_\Omega \big(\nabla u_m \nabla\xi - h\xi\big) \exp(\gamma u_m)\, dx = 0 \ . \end{aligned}$$

Note that on account of the growth condition

$$|g_{\varepsilon_m}(u,p)| \le |u|\,|p|^2$$

and the uniform bound $\|u_m\|_{L^\infty} \le C_0$ derived above, for $|\gamma| \ge C_0$ the term

$$\gamma|\nabla u_m|^2 + g_{\varepsilon_m}(u_m, \nabla u_m)$$

has the same sign as γ. Moreover, this term converges pointwise almost everywhere to the same expression involving u instead of u_m. Hence, by Fatou's lemma, upon passing to the limit $m \to \infty$ we obtain

$$\langle \xi \exp(\gamma u), A(u) - h \rangle \leq 0$$

if $\gamma \geq C_0$, respectively ≥ 0, if $\gamma \leq -C_0$. This holds for all non-negative $\xi \in C_0^\infty(\Omega)$ and hence also for $\xi \geq 0$ belonging to $H_0^{1,2} \cap L^\infty(\Omega)$. Setting $\xi = \xi_0 \exp(-\gamma u)$ we obtain

$$\langle \xi_0, A(u) - h \rangle \leq 0 \text{ and } \geq 0,$$

for any $\xi_0 \geq 0$, $\xi_0 \in C_0^\infty(\Omega)$. Hence u is a weak solution of (3.4), (3.5), as desired.

More sophisticated variants and applications of Theorem 3.4 are given by Bensoussan-Boccardo-Murat [1], Boccardo-Murat-Puel [1], and Frehse [1], [2].

Hardy Space Methods

The Hardy $\mathcal{H}^p$-spaces play an important role in harmonic analysis. In the theory of partial differential equations, the space $\mathcal{H}^1$ is of particular interest. For instance, the Calderón-Zygmund theorem, asserting that for $1 < p < \infty$ any function $u \in L^p$ with $\Delta u \in L^p$ belongs to $H^{2,p}_{loc}$, ceases to be valid in the limit case $p = 1$. However, the theorem remains true if we substitute L^1 by the slightly smaller space $\mathcal{H}^1$. While the harmonic analysts' definition of $\mathcal{H}^1$ is rather unwieldy and hardly lends itself to applications in the theory of partial differential equations, recently an observation of Müller [2], [3] has led to the discovery of simple criteria for composite functions to belong to $\mathcal{H}^1$. In particular, it was shown by Coifman-Lions-Meyer-Semmes [1] that the Jacobian of a function $u \in H^{1,n}(\mathbb{R}^n; \mathbb{R}^n)$ belongs to this space, and similarly for $l \times l$-sub-determinants of ∇u for $u \in H^{1,l}(\mathbb{R}^n; \mathbb{R}^N)$, for any $l, n, N \in \mathbb{N}$.

As an application, we derive the assertion of Theorem 3.4 in the case that for a sequence $u_m \rightharpoonup u$ weakly in $H^{1,2}(\mathbb{R}^n; \mathbb{R}^N)$ the sequence $-\Delta u_m = f_m$ is bounded in $\mathcal{H}^1$. In fact, by the extension of the Calderón-Zygmund theorem to $\mathcal{H}^1$, the sequence $(\nabla^2 u_m)$ is bounded in $\mathcal{H}^1 \subset L^1$, and therefore, by Rellich's compactness result, $\nabla u_m \to \nabla u$ locally in L^1 as $m \to \infty$. Boundedness of (f_m) in $\mathcal{H}^1$ by the result of Coifman-Lions-Meyer-Semmes [1] is guaranteed if, for instance, for each $m \in \mathbb{N}$ the function f_m is a linear combination of 2×2 sub-determinants of $\nabla F(u_m)$, where $F\colon \mathbb{R}^N \to \mathbb{R}^L$ is a smooth, bounded function with bounded derivative. For more information about Hardy spaces and their applications, see, for instance, Torchinsky [1] or Semmes [1].