Chapter I

The Direct Methods in the Calculus of Variations

Many problems in analysis can be cast into the form of functional equations $F(u) = 0$, the solution u being sought among a class of admissible functions belonging to some Banach space V.

Typically, these equations are nonlinear; for instance, if the class of admissible functions is restricted by some (nonlinear) constraint.

A particular class of functional equations is the class of Euler-Lagrange equations

$$DE(u) = 0$$

for a functional E on V, which is Fréchet-differentiable with derivative DE. We say such equations are of *variational form*.

For equations of variational form an extensive theory has been developed, and variational principles play an important role in mathematical physics and differential geometry, optimal control and numerical anlysis.

We briefly recall the basic definitions that will be needed in this and the following chapters, see Appendix C for details: Suppose E is a Fréchet-differentiable functional on a Banach space V with normed dual V^* and duality pairing $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R}$, and let $DE : V \to V^*$ denote the Fréchet-derivative of E. Then the directional (Gateaux-) derivative of E at u in the direction of v is given by

$$\frac{d}{d\varepsilon} E(u + \varepsilon v)\Big|_{\varepsilon = 0} = \langle v, DE(u) \rangle = DE(u)\, v.$$

For such E, we call a point $u \in V$ *critical* if $DE(u) = 0$; otherwise, u is called *regular*. A number $\beta \in \mathbb{R}$ is a *critical value* of E if there exists a critical point u of E with $E(u) = \beta$. Otherwise, β is called *regular*. Of particular interest (also in the non-differentiable case) will be relative minima of E, possibly subject to constraints. Recall that for a set $M \subset V$ a point $u \in M$ is an *absolute minimizer* for E on M if for all $v \in M$ there holds $E(v) \geq E(u)$. A point $u \in M$ is a *relative minimizer* for E on M if for some neighborhood U of u in V it is absolutely E-minimizing in $M \cap U$. Moreover, in the differentiable case, we shall also be interested in the existence of *saddle points*, that is, critical points u of E such that any neighborhood U of u in V contains points v, w such that $E(v) < E(u) < E(w)$. In physical systems, saddle points appear as unstable equilibria or transient excited states.

In this chapter we review some basic methods for proving the existence of relative minimizers. Somewhat imprecisely we summarily refer to these methods as the *direct methods* in the calculus of variations. However, besides the classical lower semi-continuity and compactness method we also include the compensated compactness method of Murat and Tartar, and the concentration-compactness principle of P.L. Lions. Moreover, we recall Ekeland's variational principle and the duality method of Clarke and Ekeland.

Applications will be given to problems concerning minimal hypersurfaces, semilinear and quasi-linear elliptic boundary value problems, finite elasticity, Hamiltonian systems, and semilinear wave equations.

From the beginning it will be apparent that in order to achieve a satisfactory existence theory the notion of solution will have to be suitably relaxed. Hence, in general, the above methods will at first only yield generalized or "weak" solutions of our problems. A second step often will be necessary to show that these solutions are regular enough to be admitted as classical solutions. The regularity theory in many cases is very subtle and involves a delicate machinery. It would go beyond the scope of this book to cover this topic completely. However, for the problems that we will mostly be interested in, the regularity question can be dealt with rather easily. The reader will find this material in Appendix B. References to more advanced texts on the regularity issue will be given where appropriate.

1. Lower Semi-continuity

In this section we give sufficient conditions for a functional to be bounded from below and to attain its infimum.

The discussion can be made largely independent of any differentiability assumptions on E or structure assumptions on the underlying space of admissible functions M. In fact, we have the following classical result.

1.1 Theorem. *Let M be a topological Hausdorff space, and suppose $E : M \to \mathbb{R} \cup +\infty$ satisfies the condition of bounded compactness:*

$$\text{(1.1)} \qquad \begin{array}{l} \textit{For any } \alpha \in \mathbb{R} \textit{ the set} \\ \qquad K_\alpha = \{u \in M \; ; \; E(u) \leq \alpha\} \\ \textit{is compact (Heine-Borel property).} \end{array}$$

Then E is uniformly bounded from below on M and attains its infimum. The conclusion remains valid if instead of (1.1) we suppose that any sub-level set K_α is sequentially compact.

Remark. Necessity of condition (1.1) is illustrated by simple examples: The function $E\colon [-1,1] \to \mathbb{R}$ given by $E(x) = x^2$ if $x \neq 0$, $E(x) = 1$ if $x = 0$, or the exponential function $E(x) = \exp(x)$ on $\mathbb{R}$ are bounded from below but do not admit a minimizer. Note that the space M in the first example is compact while in the second example the function E is smooth – even analytic.

Proof of Theorem 1.1. Suppose (1.1) holds. We may assume $E \not\equiv +\infty$. Let

$$\alpha_0 = \inf_M E \geq -\infty,$$

and let (α_m) be the strictly decreasing sequence

$$\alpha_m \searrow \alpha_0 \qquad (m \to \infty)\ .$$

Let $K_m = K_{\alpha_m}$. By assumption, each K_m is compact and non-empty. Moreover, $K_m \supset K_{m+1}$ for all m. By compactness of K_m there exists a point $u \in \bigcap_{m \in \mathbb{N}} K_m$, satisfying

$$E(u) \leq \alpha_m, \qquad \text{for all } m.$$

Passing to the limit $m \to \infty$ we obtain that

$$E(u) \leq \alpha_0 = \inf_M E,$$

and the claim follows.

If instead of (1.1) each K_α is sequentially compact, we choose a *minimizing sequence* (u_m) in M such that $E(u_m) \to \alpha_0$. Then for any $\alpha > \alpha_0$ the sequence (u_m) will eventually lie entirely within K_α. By sequential compactness of K_α therefore (u_m) will accumulate at a point $u \in \bigcap_{\alpha > \alpha_0} K_\alpha$ which is the desired minimizer. □

Note that if $E : M \to \mathbb{R}$ satisfies (1.1), then for any $\alpha \in \mathbb{R}$ the set

$$\{u \in M\ ;\ E(u) > \alpha\} = M \setminus K_\alpha$$

is open, that is, E is *lower semi-continuous.* (Respectively, if each K_α is sequentially compact, then E will be sequentially lower semi-continuous.) Conversely, if E is (sequentially) lower semi-continuous and for some $\overline{\alpha} \in \mathbb{R}$ the set $K_{\overline{\alpha}}$ is (sequentially) compact, then K_α will be (sequentially) compact for all $\alpha \leq \overline{\alpha}$ and again the conclusion of Theorem 1.1 will be valid.

Note that the lower semi-continuity condition can be more easily fulfilled the finer the topology on M. In contrast, the condition of compactness of the sub-level sets K_α , $\alpha \in \mathbb{R}$, calls for a coarse topology and both conditions are competing. In practice, there is often a natural weak Sobolev space topology where both conditions can be simultaneously satisfied. However, there are many interesting cases where condition (1.1) cannot hold in *any* reasonable topology (even though relative minimizers may exist). Later in this chapter we shall see some examples and some more delicate ways of handling the possible loss of compactness. See Section 4; see also Chapter III.

In applications, the conditions of the following special case of Theorem 1.1 can often be checked more easily.

1.2 Theorem. *Suppose V is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset V$ be a weakly closed subset of V. Suppose $E : M \to \mathrm{I\!R} \cup +\infty$ is coercive and (sequentially) weakly lower semi-continuous on M with respect to V, that is, suppose the following conditions are fullfilled:*
(1°) $E(u) \to \infty$ as $\|u\| \to \infty,\ u \in M$.
(2°) For any $u \in M$, any sequence (u_m) in M such that $u_m \rightharpoonup u$ weakly in V there holds:

$$E(u) \leq \liminf_{m\to\infty} E(u_m) \ .$$

Then E is bounded from below on M and attains its infimum in M.

The concept of minimizing sequences offers a direct and (apparently) constructive proof.

Proof. Let $\alpha_0 = \inf_M E$ and let (u_m) be a minimizing sequence in M, that is, satisfying $E(u_m) \to \alpha_0$. By coerciveness, (u_m) is bounded in V. Since V is reflexive, by the Eberlein-Šmulian theorem (see Dunford-Schwartz [1; p. 430]) we may assume that $u_m \rightharpoonup u$ weakly for some $u \in V$. But M is weakly closed, therefore $u \in M$, and by weak lower semi-continuity

$$E(u) \leq \liminf_{m\to\infty} E(u_m) = \alpha_0 \ . \qquad \square$$

Examples. An important example of a sequentially weakly lower semi-continous functional is the norm in a Banach space V. Closed and convex subsets of Banach spaces are important examples of weakly closed sets. If V is the dual of a separable normed vector space, Theorem 1.2 and its proof remain valid if we replace weak by weak*-convergence.

We present some simple applications.

Degenerate Elliptic Equations

1.3 Theorem. *Let Ω be a bounded domain in $\mathrm{I\!R}^n$, $p \in [2, \infty[$ with conjugate exponent q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and let $f \in H^{-1,q}(\Omega)$, the dual of $H_0^{1,p}(\Omega)$, be given. Then there exists a weak solution $u \in H_0^{1,p}(\Omega)$ of the boundary value problem*

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

in the sense that u satisfies the equation

$$\int_\Omega (\nabla u|\nabla u|^{p-2}\nabla\varphi - f\varphi)dx = 0 \ , \qquad \forall \varphi \in C_0^\infty(\Omega) \ . \tag{1.4}$$

Proof. Note that the left part of (1.4) is the directional derivative of the C^1-functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx \ - \ \int_\Omega f u \, dx$$

on the Banach space $V = H_0^{1,p}(\Omega)$ in direction φ; that is, problem (1.2), (1.3) is of variational form.

Note that $H_0^{1,p}(\Omega)$ is reflexive. Moreover, E is coercive. In fact, we have

$$\begin{aligned} E(u) &\geq \frac{1}{p}\|u\|^p_{H_0^{1,p}} \ - \ \|f\|_{H^{-1,q}} \ \|u\|_{H_0^{1,p}} \geq \frac{1}{p}\left(\|u\|^p_{H_0^{1,p}} - c\|u\|_{H_0^{1,p}}\right) \\ &\geq c^{-1}\|u\|^p_{H_0^{1,p}} \ - \ C. \end{aligned}$$

Finally, E is (sequentially) weakly lower semi-continuous: It suffices to show that for $u_m \rightharpoonup u$ weakly in $H_0^{1,p}(\Omega)$ we have

$$\int_\Omega f \, u_m \, dx \quad \to \quad \int_\Omega f \, u \, dx \ .$$

Since $f \in H^{-1,q}(\Omega)$, however, this follows from the very definition of weak convergence. Hence Theorem 1.2 is applicable and there exists a minimizer $u \in H_0^{1,p}(\Omega)$ of E, solving (1.4). □

Note that for $p \geq 2$ the p-Laplacian is strongly monotone in the sense that

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \cdot (\nabla u - \nabla v) \, dx \geq c\|u - v\|^p_{H_0^{1,p}} \ .$$

In particular, the solution u to (1.4) is unique.

If f is more regular, say $f \in C^{m,\alpha}(\overline{\Omega})$, we would expect the solution u of (1.4) to be more regular as well. This is true if $p = 2$, see Appendix B, but in the degenerate case $p > 2$, where the uniform ellipticity of the p-Laplace operator is lost at zeros of $|\nabla u|$, the best that one can hope for is $u \in C^{1,\alpha}(\overline{\Omega})$; see Uhlenbeck [1], Tolksdorf [2; p. 128], Di Benedetto [1].

In Theorem 1.3 we have applied Theorem 1.2 to a functional on a reflexive space. An example in a non-reflexive setting is given next.

Minimal Partitioning Hypersurfaces

For a domain $\Omega \subset \mathbb{R}^n$ let $BV(\Omega)$ be the space of functions $u \in L^1(\Omega)$ such that

$$\int_\Omega |Du| = \sup\Big\{ \int_\Omega \sum_{i=1}^n u D_i g_i \, dx \; ; \; g = (g_1, \ldots, g_n) \in C_0^1(\Omega; \mathbb{R}^n), \; |g| \le 1 \Big\} < \infty \, ,$$

endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + \int_\Omega |Du| \, .$$

$BV(\Omega)$ is a Banach space, embedded in $L^1(\Omega)$, and – provided Ω is bounded and sufficiently smooth – by Rellich's theorem the injection $BV(\Omega) \hookrightarrow L^1(\Omega)$ is compact; see for instance Giusti [1; Theorem 1.19, p. 17]. Moreover, the function $u \mapsto \int_\Omega |Du|$ is lower semi-continuous with respect to L^1-convergence.

Let χ_G be the characteristic function of a set $G \subset \mathbb{R}^n$; that is, $\chi_G(x) = 1$ if $x \in G$, $\chi_G(x) = 0$ else. Also let $\mathcal{L}^n$ denote the n-dimensional Lebesgue measure.

1.4 Theorem. *Let Ω be a smooth, bounded domain in $\mathbb{R}^n$. Then there exists a subset $G \subset \Omega$ such that*

$$(1°) \qquad \mathcal{L}^n(G) = \mathcal{L}^n(\Omega \setminus G) = \frac{1}{2}\mathcal{L}^n(\Omega)$$

and such that its perimeter with respect to Ω,

$$(2°) \qquad P(G, \Omega) = \int_\Omega |D\chi_G| \, ,$$

is minimal among all sets satisfying (1°).

Proof. Let $M = \{\chi_G \; ; \; G \subset \Omega \text{ is measurable and satisfies } (1°)\}$, endowed with the L^1-topology, and let $E : M \to \mathbb{R} \cup +\infty$ be given by

$$E(u) = \int_\Omega |Du| \, .$$

Since $\|\chi_G\|_{L^1} \le \mathcal{L}^n(\Omega)$, the functional E is coercive on M with respect to the norm in $BV(\Omega)$. Since bounded sets in $BV(\Omega)$ are relatively compact in $L^1(\Omega)$ and since M is closed in $L^1(\Omega)$, by weak lower semi-continuity of E in $L^1(\Omega)$ the sub-level sets of E are compact. The conclusion now follows from Theorem 1.1. □

The support of the distribution $D\chi_G$, where G has minimal perimeter (2°) with respect to Ω, can be interpreted as a minimal bisecting hypersurface, dividing Ω into two regions of equal volume. The regularity of the dividing hypersurface is intimately connected with the existence of minimal cones in $\mathbb{R}^n$. See Giusti [1] for further material on functions of bounded variation, sets of bounded perimeter, the area integrand, and applications.

A related setting for the study of minimal hypersurfaces and related objects is offered by geometric measure theory. Also in this field variational principles play an important role; see for instance Almgren [1], Morgan [1], or Simon [1] for introductory material and further references.

Our next example is concerned with a parametric approach.

Minimal Hypersurfaces in Riemannian Manifolds

Let Ω be a bounded domain in $\mathbb{R}^n$, and let S be a compact subset in $\mathbb{R}^N$. Also let $u_0 \in H^{1,2}(\Omega;\mathbb{R}^N)$ with $u_0(\Omega) \subset S$ be given. Define

$$H^{1,2}(\Omega;S) = \left\{u \in H^{1,2}(\Omega;\mathbb{R}^N)\ ;\ u(\Omega) \subset S \ \text{ almost everywhere}\right\}$$

and let

$$M = \left\{u \in H^{1,2}(\Omega;S)\ ;\ u - u_0 \in H_0^{1,2}(\Omega;\mathbb{R}^N)\right\}\ .$$

Then, by Rellich's theorem, M is closed in the weak topology of $V = H^{1,2}(\Omega;\mathbb{R}^N)$. For $u = (u^1, \ldots, u^N) \in H^{1,2}(\Omega;S)$ let

$$E(u) = \int_\Omega g_{ij}(u)\nabla u^i \nabla u^j\ dx\ ,$$

where $g = (g_{ij})_{1\le i,j\le N}$ is a given positively definite symmetric matrix with coefficients $g_{ij}(u)$ depending continuously on $u \in S$, and where, by convention, we tacitly sum over repeated indices $1 \le i,j \le N$. Note that since S is compact g is uniformly positive definite on S, and there exists $\lambda > 0$ such that $E(u) \ge \lambda\,\|\nabla u\|_{L^2}^2$ for $u \in H^{1,2}(\Omega;S)$. In addition, since S and Ω are bounded, we have that $\|u\|_{L^2} \le c$ uniformly, for $u \in H^{1,2}(\Omega;S)$. Hence E is coercive on $H^{1,2}(\Omega;S)$ with respect to the norm in $H^{1,2}(\Omega;\mathbb{R}^N)$.

Finally, E is lower semi-continuous in $H^{1,2}(\Omega;S)$ with respect to weak convergence in $H^{1,2}(\Omega;\mathbb{R}^N)$. Indeed, if $u_m \rightharpoonup u$ weakly in $H^{1,2}(\Omega;\mathbb{R}^N)$, by Rellich's theorem $u_m \to u$ strongly in L^2 and hence a subsequence (u_m) converges almost everywhere. By Egorov's theorem, given $\delta > 0$ there is an exceptional set Ω_δ of measure $\mathcal{L}^n(\Omega_\delta) < \delta$ such that $u_m \to u$ uniformly on $\Omega \setminus \Omega_\delta$. We may assume that $\Omega_\delta \subset \Omega_{\delta'}$ for $\delta \le \delta'$. By weak lower semi-continuity of the semi-norm on $H^{1,2}(\Omega;\mathbb{R}^N)$, defined by

$$|v|^2 = \int_{\Omega\setminus\Omega_\delta} g_{ij}(u)\,\nabla v^i \nabla v^j\ dx,$$

then

$$\begin{aligned}
\int_{\Omega\setminus\Omega_\delta} & g_{ij}(u)\,\nabla u^i \nabla u^j\,dx \\
&\le \liminf_{m\to\infty} \int_{\Omega\setminus\Omega_\delta} g_{ij}(u)\,\nabla u_m^i \nabla u_m^j\,dx \\
&= \liminf_{m\to\infty} \int_{\Omega\setminus\Omega_\delta} g_{ij}(u_m)\,\nabla u_m^i \nabla u_m^j\,dx \\
&\le \liminf_{m\to\infty} E(u_m)\ .
\end{aligned}$$

Passing to the limit $\delta \to 0$, from Beppo Levi's theorem we obtain

$$\begin{aligned}
E(u) &= \lim_{\delta\to 0} \int_{\Omega\setminus\Omega_\delta} g_{ij}(u)\,\nabla u^i \nabla u^j\,dx \\
&\le \liminf_{m\to\infty}\ E(u_m)\ .
\end{aligned}$$

Applying Theorem 1.2 to E on M we obtain

1.5 Theorem. *For any boundary data $u_0 \in H^{1,2}(\Omega; S)$ there exists an E-minimal extension $u \in M$.*

In differential geometry Theorem 1.5 arises in the study of harmonic maps $u : \Omega \to S$ from a domain Ω into an N-dimensional manifold S with metric g for prescribed boundary data $u = u_0$ on $\partial\Omega$. Like in the previous example, the regularity question is related to the existence of special harmonic maps; in this case, singularities of harmonic maps from Ω into S are related to harmonic mappings of spheres into S. For further references see Eells-Lemaire [1], [2], Hildebrandt [3], Jost [2]. For questions concerning regularity see Giaquinta-Giusti [1], Schoen-Uhlenbeck [1], [2].

A General Lower Semi-continuity Result

We now conclude this short list of introductory examples and return to the development of the variational theory. Note that the property of E being lower semi-continuous with respect to some weak kind of convergence is at the core of the above existence results. In Theorem 1.6 below we establish a lower semi-continuity result for a very broad class of variational integrals, including and going beyond those encountered in Theorem 1.5, as Theorem 1.6 would also apply in the case of unbounded targets S and possibly degenerate or singular metrics g.

We consider variational integrals

$$E(u) = \int_\Omega F(x, u, \nabla u)\,dx \tag{1.5}$$

involving (vector-valued) functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$.

1.6 Theorem. *Let Ω be a domain in $\mathbb{R}^n$, and assume that $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Caratheodory function satisfying the conditions*
(1°) $F(x,u,p) \geq \phi(x)$ for almost every x, u, p, where $\phi \in L^1(\Omega)$.
(2°) $F(x,u,\cdot)$ is convex in p for almost every x, u.
Then, if u_m, $u \in H^{1,1}_{loc}(\Omega)$ and $u_m \to u$ in $L^1(\Omega')$, $\nabla u_m \rightharpoonup \nabla u$ weakly in $L^1(\Omega')$ for all bounded $\Omega' \subset\subset \Omega$, it follows that

$$E(u) \leq \liminf_{m\to\infty} E(u_m) \ ,$$

where E is given by (1.5).

Notes. In the scalar case $N = 1$, weak lower semi-continuity results like Theorem 1.6 were first stated by Tonelli [1] and Morrey [1]; these results were then extended and simplified by Serrin [1], [2] who showed that for non-negative, smooth functions $F(x,u,p)\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, which are convex in p , the functional E given by (1.5) is lower semi-continuous with respect to convergence in $L^1_{loc}(\Omega)$. A corresponding result in the vector-valued case $N > 1$ subsequently was derived by Morrey [4; Theorem 4.1.1]; however, Eisen [1] not only pointed out a gap in Morrey's proof but also gave an example showing that for $N > 1$ in general, Theorem 1.6 ceases to be true without the assumption that the L^1-norms of ∇u_m are uniformly locally bounded. Theorem 1.6 is due to Berkowitz [1] and Eisen [2]. Related results can be found for instance in Morrey [4; Theorem 1.8.2], or Giaquinta [1]. Our proof is modeled on Eisen [2].

Proof. We may assume that $\big(E(u_m)\big)$ is finite and convergent. Moreover, replacing F by $F - \phi$ we may assume that $F \geq 0$. Let $\Omega' \subset\subset \Omega$ be given. By weak local L^1-convergence $\nabla u_m \rightharpoonup \nabla u$, for any $m_0 \in \mathbb{N}$ there exists a sequence $(P^l)_{l \geq m_0}$ of convex linear combinations

$$P^l = \sum_{m=m_0}^{l} \alpha^l_m \nabla u_m \ , \ 0 \leq \alpha^l_m \leq 1 \ , \ \sum_{m=m_0}^{l} \alpha^l_m = 1 \ , \ l \geq m_0$$

such that $P^l \to \nabla u$ strongly in $L^1(\Omega')$ and pointwise almost everywhere as $l \to \infty$; see for instance Rudin [1; Theorem 3.13]. By convexity, for any m_0, any $l \geq m_0$, and almost every $x \in \Omega'$:

$$\begin{aligned} F\left(x, u(x), P^l(x)\right) &= F\left(x, u(x), \sum_{m=m_0}^{l} \alpha^l_m \nabla u_m(x)\right) \\ &\leq \sum_{m=m_0}^{l} \alpha^l_m F\left(x, u(x), \nabla u_m(x)\right) \ . \end{aligned}$$

Integrating over Ω' and passing to the limit $l \to \infty$, from Fatou's lemma we obtain:

$$\int_{\Omega'} F(x,u(x),\nabla u(x))\,dx \le \liminf_{l\to\infty} \int_{\Omega'} F\big(x,u(x),P^l(x)\big)\,dx$$
$$\le \sup_{m\ge m_0} \int_{\Omega'} F\big(x,u(x),\nabla u_m(x)\big)\,dx\ .$$

Since m_0 was arbitrary, this implies that

$$\int_{\Omega'} F\big(x,u(x),\nabla u(x)\big)\,dx \le \limsup_{m\to\infty} \int_{\Omega'} F(x,u(x),\nabla u_m(x))\,dx\,,$$

for any bounded $\Omega' \subset\subset \Omega$.

Now we need the following result (Eisen [2; p. 75]).

1.7 Lemma. *Under the hypotheses of Theorem 1.6 on F, u_m, and u there exists a subsequence (u_m) such that:*

$$F(x,u_m(x),\nabla u_m(x)) - F(x,u(x),\nabla u_m(x)) \to 0$$

in measure, locally in Ω.

Proof of Theorem 1.6 (completed). By Lemma 1.7 for any $\Omega' \subset\subset \Omega$, any $\varepsilon > 0$, and any $m_0 \in \mathbb{N}$ there exists $m \ge m_0$ and a set $\Omega'_{\varepsilon,m} \subset \Omega'$ with $\mathcal{L}^n\left(\Omega'_{\varepsilon,m}\right) < \varepsilon$ such that

$$|F\big(x,u_m(x),\nabla u_m(x)\big) - F\big(x,u(x),\nabla u_m(x)\big)| < \varepsilon \tag{1.6}$$

for all $x \in \Omega' \setminus \Omega'_{\varepsilon,m}$. Replacing ε by $\varepsilon_m = 2^{-m}$ and passing to a subsequence, if necessary, we may assume that for each m there is a set $\Omega'_{\varepsilon_m,m} \subset \Omega'$ of measure $< \varepsilon_m$ such that (1.6) is satisfied (with ε_m) for all $x \in \Omega' \setminus \Omega'_{\varepsilon_m,m}$. Hence, for any given $\varepsilon > 0$, if we choose $m_0 = m_0(\varepsilon) > |\log_2 \varepsilon|$, $\Omega'_\varepsilon = \bigcup_{m\ge m_0} \Omega'_{\varepsilon_m,m}$, this set has measure $\mathcal{L}^n(\Omega'_\varepsilon) < \varepsilon$ and inequality (1.6) holds uniformly for all $x \in \Omega' \setminus \Omega'_\varepsilon$, and all $m \ge m_0(\varepsilon)$. Moreover, for $\varepsilon < \delta$ by construction $\Omega'_\varepsilon \subset \Omega'_\delta$.

Cover Ω by disjoint bounded sets $\Omega^{(k)} \subset\subset \Omega$, $k \in \mathbb{N}$. Let $\varepsilon > 0$ be given and choose a sequence $\varepsilon^{(k)} > 0$, such that $\sum_{k\in\mathbb{N}} \mathcal{L}^n\left(\Omega^{(k)}\right)\varepsilon^{(k)} \le \varepsilon$. Passing to a subsequence, if necessary, for each $\Omega^{(k)}$ and $\varepsilon^{(k)}$ we may choose $m_0^{(k)}$ and $\Omega_\varepsilon^{(k)} \subset \Omega^{(k)}$ such that $\mathcal{L}^n\left(\Omega_\varepsilon^{(k)}\right) < \varepsilon^{(k)}$ and

$$|F(x,u_m(x),\nabla u_m(x)) - F(x,u(x),\nabla u_m(x))| < \varepsilon^{(k)}$$

uniformly for $x \in \Omega^{(k)} \setminus \Omega_\varepsilon^{(k)}$, $m \ge m_0^{(k)}$. Moreover, we may assume that $\Omega_\varepsilon^{(k)} \subset \Omega_\delta^{(k)}$, if $\varepsilon < \delta$, for all k. Then for any $K \in \mathbb{N}$, letting $\Omega^K = \cup_{k=1}^K \Omega^{(k)}$, $\Omega_\varepsilon^K = \cup_{k=1}^K \Omega_\varepsilon^{(k)}$, we have

$$\begin{aligned}\int_{\Omega^K \setminus \Omega^K_\varepsilon} & F(x,u,\nabla u)\,dx \\ & \leq \limsup_{m\to\infty} \int_{\Omega^K \setminus \Omega^K_\varepsilon} F\left(x,u,\nabla u_m\right)\,dx \\ & \leq \limsup_{m\to\infty} \int_{\Omega^K \setminus \Omega^K_\varepsilon} F\left(x,u_m,\nabla u_m\right)\,dx + \varepsilon \\ & \leq \limsup_{m\to\infty} E(u_m) + \varepsilon = \liminf_{m\to\infty} E(u_m) + \varepsilon \ .\end{aligned}$$

Letting $\varepsilon \to 0$ and then $K \to \infty$, the claim follows from Beppo Levi's theorem, since $F \geq 0$ and since $\Omega^K \setminus \Omega^K_\varepsilon$ increases when $\varepsilon \downarrow 0$, followed by $K \uparrow \infty$. □

Proof of Lemma 1.7. We basically follow Eisen [2]. Suppose by contradiction that there exist $\Omega' \subset\subset \Omega$ and $\varepsilon > 0$ such that, letting

$$\Omega_m = \{x \in \Omega' \ ; \ |F\left(x,u_m,\nabla u_m\right) - F\left(x,u,\nabla u_m\right)| \geq \varepsilon\} \ ,$$

there holds

$$\liminf_{m\to\infty} \mathcal{L}^n(\Omega_m) \geq 2\varepsilon \ .$$

The sequence (∇u_m), being weakly convergent, is uniformly bounded in $L^1(\Omega')$. In particular,

$$\mathcal{L}^n\{x \in \Omega' \ ; \ |\nabla u_m(x)| \geq l\} \leq l^{-1} \int_{\Omega'} |\nabla u_m|\,dx \leq \frac{C}{l} \leq \varepsilon \ ,$$

if $l \geq l_0(\varepsilon)$ is large enough. Setting $\tilde{\Omega}_m := \{x \in \Omega_m \ ; \ |\nabla u_m(x)| \leq l_0(\varepsilon)\}$ therefore there holds

$$\liminf_{m\to\infty} \mathcal{L}^n\left(\tilde{\Omega}_m\right) \geq \varepsilon.$$

Hence also for $\Omega^M = \bigcup_{m\geq M} \tilde{\Omega}_m$ we have

$$\mathcal{L}^n(\Omega^M) \geq \varepsilon \ ,$$

uniformly in $M \in \mathbb{N}$. Moreover, $\Omega' \supset \Omega^M \supset \Omega^{M+1}$ for all M and therefore $\Omega^\infty := \bigcap_{M\in\mathbb{N}} \Omega^M \subset \Omega'$ has $\mathcal{L}^n(\Omega^\infty) \geq \varepsilon$. Finally, neglecting a set of measure zero and passing to a subsequence, if necessary, we may assume that $F(x,z,p)$ is continuous in (z,p), that $u_m(x)$, $u(x)$, $\nabla u_m(x)$ are unambiguously defined and finite while $u_m(x) \to u(x)$ as $m \to \infty$ at every point $x \in \Omega^\infty$.

Note that every point $x \in \Omega^\infty$ by construction belongs to infinitely many of the sets $\tilde{\Omega}_m$. Choose such a point x. Relabeling, we may assume $x \in \bigcap_{m\in\mathbb{N}} \tilde{\Omega}_m$.

By uniform boundedness $|\nabla u_m(x)| \le C$ there exists a subsequence $m \to \infty$ and a vector $p \in \mathbb{R}^{nN}$ such that $\nabla u_m(x) \to p \ \ (m \to \infty)$. But then by continuity

$$F\big(x, u_m(x), \nabla u_m(x)\big) \to F\big(x, u(x), p\big)$$

while also

$$F\big(x, u(x), \nabla u_m(x)\big) \to F\big(x, u(x), p\big)$$

which contradicts the characterization of Ω_m given above. □

1.8 Remarks. The following observations may be useful in applications.
(1°) Theorem 1.6 also applies to functionals involving higher (mth-) order derivatives of a function u by letting $U = (u, \nabla u, \dots, \nabla^{m-1} u)$ denote the $(m-1)$-jet of u. Note that convexity is only required in the highest-order derivatives $P = \nabla^m u$.
(2°) If (u_m) is bounded in $H^{1,1}(\Omega')$ for any $\Omega' \subset\subset \Omega$, by Rellich's theorem and repeated selection of subsequences there exists a subsequence (u_m) which converges strongly in $L^1(\Omega')$ for any $\Omega' \subset\subset \Omega$.

Local boundedness in $H^{1,1}$ of a minimizing sequence (u_m) for E can be inferred from a coerciveness condition like

$$F(x, z, p) \ge |p|^\mu - \phi(x), \ \ \mu \ge 1, \ \ \phi \in L^1 \ . \tag{1.7}$$

The delicate part in the hypotheses concerning (u_m) is the assumption that (∇u_m) converges weakly in L^1_{loc}. In case $\mu > 1$ in (1.7) this is clear, but in case $\mu = 1$ the local L^1-limit of a minimizing sequence may lie in BV_{loc} instead of $H^{1,1}_{loc}$. See Theorem 1.4, for example; see also Section 3.
(3°) By convexity in p, continuity of F in (u, p) for almost every x is equivalent to the following condition, which is easier to check in applications:

> $F(x, \cdot, \cdot)$ is continuous, separately in $u \in \mathbb{R}^N$ and $p \in \mathbb{R}^{nN}$, for almost every $x \in \Omega$.

Indeed, for any fixed x, u, p and all $e \in \mathbb{R}^{nN}, |e| = 1, \ \alpha \in [0,1]$, letting $q = p + \alpha e, \ p_+ = p + e, \ p_- = p - e$ and writing $F(x, u, p) = F(u, p)$ for brevity, by convexity we have

$$\begin{aligned} F(u, q) &= F(u, \alpha p_+ + (1-\alpha) p) \le \alpha F(u, p_+) + (1-\alpha) F(u, p) \ , \\ F(u, p) &= F(u, \frac{1}{1+\alpha} q + \frac{\alpha}{1+\alpha} p_-) \le \frac{1}{1+\alpha} F(u, q) + \frac{\alpha}{1+\alpha} F(u, p_-) \ . \end{aligned}$$

Hence

$$\alpha\left(F(u,p) - F(u,p_+)\right) \le F(u,p) - F(u,q) \le \alpha\left(F(u,p_-) - F(u,p)\right)$$

and it follows that

$$\sup_{|q-p| \le 1} \frac{|F(u,q) - F(u,p)|}{|q-p|} \le \sup_{|q-p|=1} |F(u,q) - F(u,p)| \ .$$

Since the sphere of radius 1 around p lies in the convex hull of finitely many vectors $q_0, q_1, \ldots, q_{nN}$, by continuity of F in u and convexity in p the right-hand side of this inequality remains uniformly bounded in a neighborhood of (u, p). Hence $F(\cdot, \cdot)$ is locally Lipschitz continous in p, locally uniformly in $(u, p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Therefore, if $u_m \to u$, $p_m \to p$ we have

$$\begin{aligned} |F(u_m, p_m) - F(u, p)| &\le |F(u_m, p_m) - F(u_m, p)| + |F(u_m, p) - F(u, p)| \\ &\le c|p_m - p| + o(1) \to 0 \qquad \text{as } m \to \infty, \end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$, as desired.
(4°) In the scalar case ($N = 1$), if F is C^2 for example, the existence of a minimizer u for E implies that the *Legendre condition*

$$\sum_{\alpha,\beta=1}^{n} F_{p_\alpha p_\beta}(x, u, p)\, \xi_\alpha \xi_\beta \ge 0, \qquad \text{for all } \xi \in \mathbb{R}^n$$

holds at all points $(x, u = u(x), p = \nabla u(x))$, see for instance Giaquinta [1; p. 11 f.]. This condition in turn implies the convexity of F in p.

The situation is quite different in the vector-valued case $N > 1$. In this case, in general only the *Legendre-Hadamard condition*

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1} F_{p^i_\alpha p^j_\beta}(x, u, p) \xi_\alpha \xi_\beta \eta^i \eta^j \ge 0 , \qquad \text{for all } \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^N$$

will hold at a minimizer, which is much weaker then convexity (Giaquinta [1; p. 12]).

In fact, in Section 3 below we shall see how, under certain additional structure conditions on F, the convexity assumption in Theorem 1.6 can be weakened in the vector-valued case.

2. Constraints

Applying the direct methods often involves a delicate interplay between the functional E, the space of admissible functions M, and the topology on M. In this section we will see how, by means of imposing constraints on admissible functions and/or by a suitable modification of the variational problem, the direct methods can be successfully employed also in situations where their use seems highly unlikely at first.

Note that we will not consider constraints that are dictated by the problems themselves, such as physical restrictions on the response of a mechanical system. Constraints of this type in general lead to variational inequalities, and we refer to Kinderlehrer-Stampacchia [1] for a comprehensive introduction to this field. Instead, we will show how certain variational problems can be solved by adding virtual – that is, purely technical – constraints to the conditions defining the admissible set, thus singling out distinguished solutions.

Semilinear Elliptic Boundary Value Problems

We start by deriving the existence of positive solutions to non-coercive, semilinear elliptic boundary value problems by a constrained minimization method. Such problems are motivated by studies of flame propagation (see for example Gel'fand [1; (15.5), p. 357]) or arise in the context of the Yamabe problem (see Section III.4).

Let Ω be a smooth, bounded domain in $\mathbb{R}^n$, and let $p > 2$. If $n \geq 3$ we also assume that p satisfies the condition $p < 2^* = \frac{2n}{n-2}$. For $\lambda \in \mathbb{R}$ consider the problem

$$-\Delta u + \lambda u = u|u|^{p-2} \quad \textit{in } \Omega\ , \tag{2.1}$$

$$u > 0 \quad \textit{in } \Omega\ , \tag{2.2}$$

$$u = 0 \quad \textit{on } \partial\Omega\ . \tag{2.3}$$

Also let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ denote the eigenvalues of the operator $-\Delta$ on $H_0^{1,2}(\Omega)$. Then we have the following result:

2.1 Theorem. *For any $\lambda > -\lambda_1$ there exists a positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to problem (2.1)–(2.3).*

Proof. Observe that Equation (2.1) is the Euler-Lagrange equation of the functional

$$\tilde{E}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda |u|^2\right) dx - \frac{1}{p}\int_\Omega |u|^p \, dx$$

on $H_0^{1,2}(\Omega)$, which is neither bounded from above nor from below on this space. However, using the homogeneity of (2.1) a solution of problem (2.1)–(2.3) can also be obtained by solving a constrained minimization problem for the functional

$$E(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda |u|^2\right) dx$$

on the Hilbert space $H_0^{1,2}(\Omega)$, restricted to the set

$$M = \{u \in H_0^{1,2}(\Omega)\ ;\ \int_\Omega |u|^p \ dx = 1\}\ .$$

We verify that $E : M \to \mathbb{R}$ satisfies the hypotheses of Theorem 1.2. By the Rellich-Kondrakov theorem the injection $H_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ is completely continuous for $p < 2^*$, if $n \geq 3$, respectively for any $p < \infty$, if $n = 1, 2$; see Theorem A.5 of Appendix A. Hence M is weakly closed in $H_0^{1,2}(\Omega)$.

Recall the Rayleigh-Ritz characterization

$$\lambda_1 = \inf_{\substack{u \in H_0^{1,2}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx} \tag{2.4}$$

of the smallest Dirichlet eigenvalue. This gives the estimate

$$E(u) \geq \frac{1}{2} \min\{1, \big(1 + \frac{\lambda}{\lambda_1}\big)\} \|u\|^2_{H_0^{1,2}} \ . \tag{2.5}$$

From this, coerciveness of E for $\lambda > -\lambda_1$ is immediate.

Weak lower semi-continuity of E follows from weak lower semi-continuity of the norm in $H_0^{1,2}(\Omega)$ and the Rellich-Kondrakov theorem. By Theorem 1.2 therefore E attains its infimum at a point $\underline{u}$ in M. Note that since $E(u) = E(|u|)$ we may assume that $\underline{u} \geq 0$.

To derive the variational equation for E first note that E is continuously Fréchet-differentiable in $H_0^{1,2}(\Omega)$ with

$$\langle v, DE(u) \rangle = \int_\Omega \big(\nabla u \nabla v + \lambda u v\big) \, dx \ .$$

Moreover, letting

$$G(u) = \int_\Omega |u|^p \, dx - 1 \ ,$$

for $p \leq 2^*$ also $G : H_0^{1,2}(\Omega) \to \mathbb{R}$ is continuously Fréchet-differentiable with

$$\langle v, DG(u) \rangle = p \int_\Omega u|u|^{p-2} v \, dx \ .$$

In particular, at any point $u \in M$

$$\langle u, DG(u) \rangle = p \int_\Omega |u|^p \, dx = p \neq 0 \ ,$$

and by the implicit function theorem the set $M = G^{-1}(0)$ is a C^1-submanifold of $H_0^{1,2}(\Omega)$.

Now, by the Lagrange multiplier rule, there exists a parameter $\mu \in \mathbb{R}$ such that

$$\begin{aligned} \langle v, (DE(\underline{u}) - \mu DG(\underline{u})) \rangle &= \int_\Omega \big(\nabla \underline{u} \nabla v + \lambda \underline{u} v - \mu \underline{u} |\underline{u}|^{p-2} v\big) \, dx \\ &= 0, \qquad \text{for all } v \in H_0^{1,2}(\Omega) \ . \end{aligned}$$

Inserting $v = \underline{u}$ into this equation yields that

$$2E(\underline{u}) = \int_\Omega \big(|\nabla \underline{u}|^2 + \lambda |\underline{u}|^2\big) \, dx = \mu \int_\Omega |\underline{u}|^p \, dx = \mu \ .$$

Since $\underline{u} \in M$ cannot vanish identically, from (2.5) we infer that $\mu > 0$. Scaling with a suitable power of μ, we obtain a weak solution $u = \mu^{\frac{1}{p-2}} \cdot \underline{u} \in H_0^{1,2}(\Omega)$ of (2.1), (2.3) in the sense that

$$(2.6) \qquad \int_\Omega \left(\nabla u \nabla v + \lambda u v - u|u|^{p-2} v\right) dx = 0 \ , \qquad \text{for all } v \in H_0^{1,2}(\Omega) \ .$$

Moreover, (2.2) holds in the weak sense $u \geq 0,\ u \not\equiv 0$. To finish the proof we use the regularity result Lemma B.3 of Appendix B and the observations following it to obtain that $u \in C^2(\Omega)$. Finally, by the strong maximum principle $u > 0$ in Ω ; see Theorem B.4. □

Observe that, at least for the kind of nonlinear problems considered here, by Lemma B.3 of Appendix B the regularity theory is taken care of and in the following we may concentrate on proving existence of (weak) solutions. However, additional structure conditions may imply further useful properties of suitable solutions. An example is symmetry.

2.2 Symmetry. By a result of Gidas-Ni-Nirenberg [1; Theorem 2.1, p. 216, and Theorem 1, p. 209], if Ω is convex and symmetric with respect to a hyperplane, say $x_1 = 0$, any positive solution u of (2.1), (2.3) is even in x_1, that is, $u(x_1, x') = u(-x_1, x')$ for all $x = (x_1, x') \in \Omega$, and $\frac{\partial u}{\partial x_1} < 0$ at any point $x = (x_1, x') \in \Omega$ with $x_1 > 0$. In particular, if Ω is a ball, any positive solution u is radially symmetric. The proof of this result uses a variant of the Alexandrov-Hopf reflection principle and the maximum principle. This method lends itself to numerous applications in many different contexts; in Section III.4 we shall see that it is even possible to derive a-priori bounds from this method in the setting of a parabolic equation on the sphere.

Perron's Method in a Variational Guise

In the previous example the constraint built into the definition of M had the effect of making the restricted functional $E = \tilde{E}|_M$ coercive. Moreover, this constraint only changed the Euler-Lagrange equations by a factor which could be scaled away using the homogeneity of the right-hand side of (2.1).

In the next application we will see that sometimes also inequality constraints can be imposed without changing the Euler-Lagrange equations at a minimizer.

2.3 Weak sub- and super-solutions. Suppose Ω is a smooth, bounded domain in $\mathbb{R}^n$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with the property that $|g(x,u)| \leq C(R)$ for any $R > 0$ and all u such that $|u(x)| \leq R$ almost everywhere. Given $u_0 \in H_0^{1,2}(\Omega)$, we then consider the equation

$$(2.7) \qquad -\Delta u = g(\cdot, u) \qquad \text{in } \Omega \ ,$$

$$(2.8) \qquad u = u_0 \qquad \text{on } \partial\Omega \ .$$

By definition $u \in H^{1,2}(\Omega)$ is a (weak) *sub-solution* to (2.7–2.8) if $u \le u_0$ on $\partial\Omega$ and

$$\int_\Omega \nabla u \nabla \varphi \, dx - \int_\Omega g(\,\cdot\,, u)\varphi \, dx \le 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega)\ ,\ \varphi \ge 0\ .$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) *super-solution* to (2.7–2.8) if in the above the reverse inequalities hold.

2.4 Theorem. *Suppose $\underline{u} \in H^{1,2}(\Omega)$ is a sub-solution while $\overline{u} \in H^{1,2}(\Omega)$ is a super-solution to problem (2.7–2.8) and assume that with constants $\underline{c}, \overline{c} \in \mathbb{R}$ there holds $-\infty < \underline{c} \le \underline{u} \le \overline{u} \le \overline{c} < \infty$, almost everywhere in Ω. Then there exists a weak solution $u \in H^{1,2}(\Omega)$ of (2.7–2.8), satisfying the condition $\underline{u} \le u \le \overline{u}$ almost everywhere in Ω.*

Proof. With no loss of generality we may assume $u_0 = 0$. Let $G(x,u) = \int_0^u g(x,v)\,dv$ denote a primitive of g. Note that (2.7–2.8) formally are the Euler-Lagrange equations of the functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(x,u)\, dx\ .$$

However, our assumptions do not allow the conclusion that E is finite or even differentiable on $V := H_0^{1,2}(\Omega)$ – the smallest space where we have any chance of verifying coerciveness. Instead we restrict E to

$$M = \big\{u \in H_0^{1,2}(\Omega)\ ;\ \underline{u} \le u \le \overline{u} \text{ almost everywhere}\big\}\ .$$

Since $\underline{u}, \overline{u} \in L^\infty$ by assumption, also $M \subset L^\infty$ and $G\big(x, u(x)\big) \le c$ for all $u \in M$ and almost every $x \in \Omega$.

Now we can verify the hypotheses of Theorem 1.2: Clearly, $V = H_0^{1,2}(\Omega)$ is reflexive. Moreover, M is closed and convex, hence weakly closed. Since M is essentially bounded, our functional $E(u) \ge \frac{1}{2}\|u\|^2_{H_0^{1,2}(\Omega)} - c$ is coercive on M. Finally, to see that E is weakly lower semi-continuous on M, it suffices to show that

$$\int_\Omega G(x,u_m)\, dx \to \int_\Omega G(x,u)\, dx$$

if $u_m \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega)$, where $u_m, u \in M$. But – passing to a subsequence, if necessary – we may assume that $u_m \to u$ pointwise almost everywhere; moreover, $|G\big(x, u_m(x)\big)| \le c$ uniformly. Hence we may appeal to Lebesgue's theorem on dominated convergence.

From Theorem 1.2 we infer the existence of a relative minimizer $u \in M$. To see that u weakly solves (2.7), for $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$ let $v_\varepsilon = \min\big\{\overline{u}, \max\{\underline{u}, u + \varepsilon\varphi\}\big\} = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon \in M$ with

$$\varphi^\varepsilon = \max\big\{0, u + \varepsilon\varphi - \overline{u}\big\} \ge 0\ ,$$
$$\varphi_\varepsilon = \max\big\{0, \underline{u} - (u + \varepsilon\varphi)\big\} \ge 0\ .$$

Note that $\varphi_\varepsilon, \varphi^\varepsilon \in H_0^{1,2} \cap L^\infty(\Omega)$.

E is differentiable in direction $v_\varepsilon - u$. Since u minimizes E in M we have

$$0 \le \langle (v_\varepsilon - u), DE(u)\rangle = \varepsilon\langle \varphi, DE(u)\rangle - \langle \varphi^\varepsilon, DE(u)\rangle + \langle \varphi_\varepsilon, DE(u)\rangle \ ,$$

so that

$$\langle \varphi, DE(u)\rangle \ge \frac{1}{\varepsilon}\big[\langle \varphi^\varepsilon, DE(u)\rangle - \langle \varphi_\varepsilon, DE(u)\rangle\big] \ .$$

Now, since $\overline{u}$ is a supersolution to (2.7), we have

$$\begin{aligned}
\langle \varphi^\varepsilon, DE(u)\rangle &= \langle \varphi^\varepsilon, DE(\overline{u})\rangle + \langle \varphi^\varepsilon, DE(u) - DE(\overline{u})\rangle \\
&\ge \langle \varphi^\varepsilon, DE(u) - DE(\overline{u})\rangle \\
&= \int_{\Omega_\varepsilon} \big\{ \nabla(u-\overline{u})\nabla(u+\varepsilon\varphi-\overline{u}) \\
&\qquad - \big(g(x,u) - g(x,\overline{u})\big)(u+\varepsilon\varphi-\overline{u})\big\}\ dx \\
&\ge \varepsilon \int_{\Omega_\varepsilon} \nabla(u-\overline{u})\nabla\varphi\ dx - \varepsilon\int_{\Omega_\varepsilon} \big|g(x,u)-g(x,\overline{u})\big|\ |\varphi|\ dx \ ,
\end{aligned}$$

where $\Omega^\varepsilon = \big\{x \in \Omega\ ;\ u(x) + \varepsilon\varphi(x) \ge \overline{u}(x) > u(x)\big\}$. Note that $\mathcal{L}^n(\Omega^\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence by absolute continuity of the Lebesgue integral we obtain that

$$\langle \varphi^\varepsilon, DE(u)\rangle \ge o(\varepsilon) \ ,$$

where $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$. Similarly, we conclude that

$$\langle \varphi_\varepsilon, DE(u)\rangle \le o(\varepsilon) \ ,$$

whence

$$\langle \varphi, DE(u)\rangle \ge 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Reversing the sign of φ and since $C_0^\infty(\Omega)$ is dense in $H_0^{1,2}(\Omega)$ we finally see that $DE(u) = 0$, as claimed. □

2.5 A special case. Let Ω be a smooth bounded domain in $\mathbb{R}^n, n \ge 3$, and let

$$g(x,u) = k(x)u - u|u|^{p-2} \ , \tag{2.9}$$

where $p = \frac{2n}{n-2}$, and where k is a continuous function such that

$$1 \le k(x) \le K < \infty$$

uniformly in Ω. Suppose $u_0 \in C^1(\overline{\Omega})$ satisfies $u_0 \ge 1$ on $\partial\Omega$.
Then $\underline{u} \equiv 1$ is a sub-solution while $\overline{u} \equiv c$ for large $c > 1$ is a super-solution to Equations (2.7)–(2.8). Consequently, (2.7)–(2.8) admits a solution $u \ge 1$.

2.6 Remark. The sub-super-solution method can also be applied to equations on manifolds. In the context of the Yamabe problem it has been used by Loewner-Nirenberg [1] and Kazdan-Warner [1]; see Section III.4. The non-linear term in this case is precisely (2.9).

The Classical Plateau Problem

One of the great successes of the direct methods in the calculus of variations was the solution of Plateau's problem for minimal surfaces.

Let Γ be a smooth Jordan curve in $\mathbb{R}^3$. From his famous experiments with soap films Plateau became convinced that any such curve is spanned by a (not necessarily unique) surface of least area.

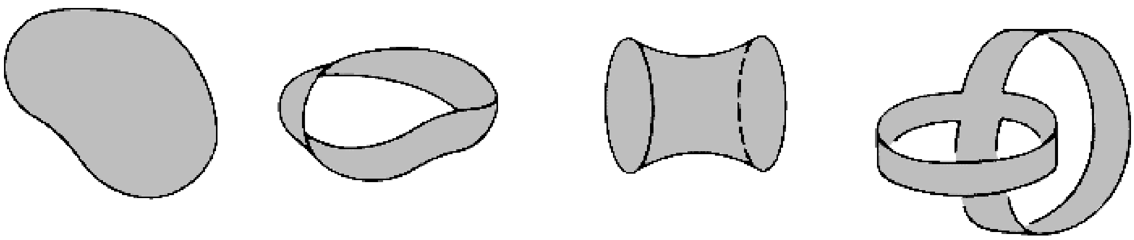

Fig. 2.1. Minimal surfaces of various topological types (disk, Möbius band, annulus, torus)

In the classical mathematical model the topological type of the surface is specified to be that of the disk

$$\Omega = \{z = (x, y)\ ;\ x^2 + y^2 < 1\}\ .$$

A naive approach to Plateau's conjecture would be to attempt to minimize the area

$$A(u) = \int_\Omega \sqrt{\det(\nabla u^t \nabla u)}\, dz = \int_\Omega \sqrt{|u_x|^2 |u_y|^2 - (u_x \cdot u_y)^2}\, dz$$

among "surfaces" $u \in H^{1,2} \cap C^0(\overline{\Omega}, \mathbb{R}^3)$ satisfying the Plateau boundary condition

(2.10) $\quad u\big|_{\partial\Omega} : \partial\Omega \to \Gamma \quad$ is a (weakly) monotone parametrization preserving the given orientation of Γ.

However, A is invariant under arbitrary changes of parameter. Hence there is no chance of achieving bounded compactness in the original variational problem and some work was necessary in order to recast this problem in a way which is accessible by direct methods. Without entering into details let us briefly report the main ideas.

It had already beeen observed by Lagrange that if a (smooth) surface S is (locally) area-minimizing for fixed boundary Γ, necessarily the mean curvature of S vanishes. In isothermal coordinates $u(x, y)$ on S this amounts to the equation

(2.11) $$\Delta u = 0\,.$$

(See Nitsche [2] or Osserman [1].) Moreover, our choice of parameter implies the conformality relations

$$|u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y \text{ in } \Omega\,, \tag{2.12}$$

in addition to the Plateau boundary condition (2.10).

We now take equations (2.10)–(2.12) as a definition for a minimal surface spanning Γ.

2.7 The variational problem. In their 1930 break-through papers, Douglas [1] and Radó [1] ingeniously proposed to solve (2.10)–(2.12) by minimizing Dirichlet's integral

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dz = \frac{1}{2}\int_\Omega (|u_x|^2 + |u_y|^2)\,dz$$

over the class

$$C(\Gamma) = \{u \in H^{1,2}(\Omega, \mathbb{R}^3)\ ;\ u\Big|_{\partial\Omega} \in C^0(\partial\Omega, \mathbb{R}^3) \text{ satisfies (2.10)}\}\ .$$

It is easy to see that

$$E(u) \geq A(u)\ ,$$

and equality holds if and only if u is conformal. Actually, we have

$$\inf_{C(\Gamma)} A(u) = \inf_{C(\Gamma)} E(u)\ .$$

This can be derived for instance from Morrey's "ε-conformality lemma" (Morrey [2; Theorem 1.2]). In Struwe [18; Appendix A] also a direct (variational) proof is given. Thus, a minimizer of E also will minimize A – hence it will satisfy (2.12) and solve the original minimization problem.

The solution of Plateau's problem is therefore reduced to the following theorem.

2.8 Theorem. *For any C^1-embedded curve Γ there exists a minimizer u of Dirichlet's integral E in $C(\Gamma)$.*

Note that $C(\Gamma) \neq \emptyset$ if $\Gamma \in C^1$. (Actually it suffices to assume that Γ is a rectifiable Jordan curve; see Douglas [1], Radó [1].)

To show Theorem 2.8, observe that in replacing A by E we have succeeded in reducing the symmetries of the problem drastically. However, E is still conformally invariant, that is

$$E(u) = E(u \circ g)$$

for all $g \in \mathcal{G}$, where

$$\mathcal{G} = \left\{g : z \mapsto g(z) = e^{i\phi}\frac{a+z}{1-\overline{a}z}\ ;\ a \in \mathbb{C},\ |a| < 1,\ 0 \leq \phi < 2\pi\right\}$$

denotes the conformal group of Möbius transformations of the disc, viewed as a subset of $\mathbb{C}$. The action of $\mathcal{G}$ is non-compact in the sense that for any $u \in C(\Gamma)$ the orbit $\{u \circ g\,;\ g \in \mathcal{G}\}$ weakly accumulates also at constant functions; see for instance Struwe [17; Lemma I.4.1] for a detailed proof. Hence $C(\Gamma)$ cannot be weakly closed in $H^{1,2}(\Omega, \mathbb{R}^3)$ and Theorem 1.2 cannot yet be applied.

Fortunately, we can also get rid of conformal invariance of E. Note that for any oriented triple $e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3} \in \partial\Omega$, $0 \le \phi_1 < \phi_2 < \phi_3 < 2\pi$ there exists a unique $g \in \mathcal{G}$ such that $g\left(e^{\frac{2\pi i k}{3}}\right) = e^{i\phi_k}$, $k = 1, 2, 3$.

Fix a parametrization γ of Γ (we may assume that γ is a C^1-diffeomorphism $\gamma : \partial\Omega \to \Gamma$) and let

$$C^*(\Gamma) = \left\{u \in C(\Gamma)\ ;\ u\left(e^{\frac{2\pi i k}{3}}\right) = \gamma\left(e^{\frac{2\pi i k}{3}}\right),\ k = 1, 2, 3\right\}\ ,$$

endowed with the $H^{1,2}$-topology. This is our space of admissible functions, normalized with respect to $\mathcal{G}$. Note that for any $u \in C(\Gamma)$ there is $g \in \mathcal{G}$ such that $u \circ g \in C^*(\Gamma)$.

The following result now is a consequence of the classical "Courant-Lebesgue lemma".

2.9 Lemma. *The set $C^*(\Gamma)$ is weakly closed in $H^{1,2}$.*

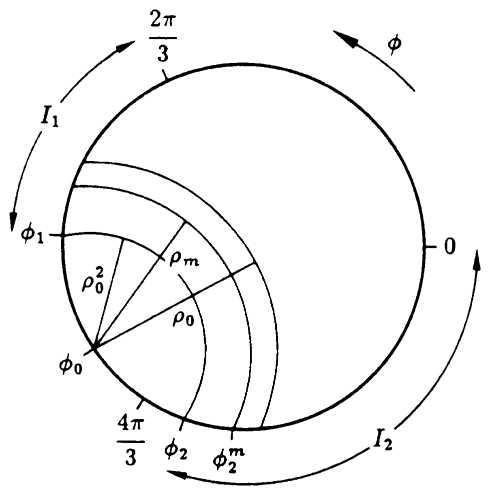

Fig. 2.2.

Proof. The proof in a subtle way uses a convexity argument as in the preceding example. To present this argument explicitly we use the fixed parametrization γ to associate with any $u \in C^*(\Gamma)$ a continuous map $\xi : \mathbb{R} \to \mathbb{R}$, such that

$$\gamma\left(e^{i\xi(\phi)}\right) = u\left(e^{i\phi}\right)\ ,\ \xi(0) = 0\ .$$

By (2.10) the functions ξ obtained in this manner are continuous, monotone and $\xi - id$ is 2π-periodic; moreover, $\xi\left(\frac{2\pi k}{3}\right) = \frac{2\pi k}{3}$, for all $k \in \mathbb{Z}$ by our three-point normalization.

Now let

$$M = \Big\{\xi : \mathbb{R} \to \mathbb{R}\ ;\ \xi \text{ is continuous and monotone,}$$
$$\xi(\phi + 2\pi) = \xi(\phi) + 2\pi,\ \xi\left(\frac{2\pi k}{3}\right) = \frac{2\pi k}{3},\ \text{for all } \phi \in \mathbb{R}, k \in \mathbb{Z}\Big\}\ .$$

Note that M is convex. Let (u_m) be a sequence in $C^*(\Gamma)$ with associated functions $\xi_m \in M$, and suppose $u_m \rightharpoonup u$ weakly in $H^{1,2}(\Omega)$. Since each ξ_m is monotone and satisfies the estimate $0 \le \xi_m(\phi) \le 2\pi$, for all $\phi \in [0, 2\pi]$, the family (ξ_m) is bounded in $BV\big([0, 2\pi]\big)$. Hence (a subsequence) $\xi_m \to \xi$ almost everywhere on $[0, 2\pi]$ and therefore – by periodicity – almost everywhere on $\mathbb{R}$, where ξ is monotone, $\xi - id$ is 2π-periodic, and ξ satisfies $\xi\left(\frac{2\pi k}{3}\right) = \frac{2\pi k}{3}$, for all $k \in \mathbb{Z}$.

Now if ξ is continuous, it follows from monotonicity that $\xi_m \to \xi$ uniformly. Thus, by continuity of γ, also u_m converges uniformly to u on $\partial\Omega$, and it follows that $u|_{\partial\Omega}$ is continuous and satisfies (2.10). That is, $u \in C^*(\Gamma)$, and the proof is complete in this case.

In order to exclude the remaining case, assume by contradiction that ξ is discontinuous at some point ϕ_0. We choose $k \in \mathbb{Z}$ such that $|\phi_0 - \frac{2\pi k}{3}| \le \frac{\pi}{3}$ and let $\left]\frac{2\pi(k-1)}{3}, \frac{2\pi(k+1)}{3}\right[=: I_0$. By monotonicity, for almost every $\phi_1, \phi_2 \in I_0$ such that $\phi_1 < \phi_0 < \phi_2$ we have

$$\frac{2\pi(k-1)}{3} \le \lim_{m\to\infty} \xi_m(\phi_1) = \xi(\phi_1) \le \lim_{\phi\to\phi_0^-} \xi(\phi)$$
$$< \lim_{\phi\to\phi_0^+} \xi(\phi) \le \xi(\phi_2) = \lim_{m\to\infty} \xi_m(\phi_2) \le \frac{2\pi(k+1)}{3}\ .$$

For such $\phi_1,\ \phi_2 \in I_0$ denote $I_1 = \{\phi \in I_0\ ;\ \phi \le \phi_1\}$, $I_2 = \{\phi \in I_0\ ;\ \phi \ge \phi_2\}$. Then by monotonicity of ξ_m and using the fact that γ is a diffeomorphism we obtain

$$\limsup_{m\to\infty} \left(\inf_{\phi\in I_1,\ \psi\in I_2} |\gamma\left(e^{i\xi_m(\phi)}\right) - \gamma\left(e^{i\xi_m(\psi)}\right)| \right)$$
$$\ge \inf_{\phi,\psi\in I_0, \phi<\phi_0<\psi} |\gamma\left(e^{i\xi(\phi)}\right) - \gamma\left(e^{i\xi(\psi)}\right)| > 0\ .$$

In particular, there exists $\varepsilon > 0$ independent of ϕ_1, ϕ_2 such that

$$|u_m(e^{i\phi}) - u_m(e^{i\psi})| \ge \varepsilon > 0 \qquad (2.13)$$

for all $\phi \in I_1,\ \psi \in I_2$ if $m \ge m_0(\phi_1, \phi_2)$ is sufficiently large.

Now let $z_0 = e^{i\phi_0}$ and for $\rho > 0$ denote

$$U_\rho = \{z \in \Omega\ ;\ |z - z_0| < \rho\},\ C_\rho = \{z \in \overline{\Omega}\ ;\ |z - z_0| = \rho\}\ .$$

Note that for all $\rho < 1$ any point $z = e^{i\phi} \in C_\rho \cap \partial\Omega$ satisfies $\phi \in I_0$.

Following Courant [1; p. 103], we will use uniform boundedness of (u_m) in $H^{1,2}$ to show that for suitable numbers $\rho_0 \in]0, 1[$, $\rho_m \in [\rho_0^2, \rho_0]$ the oscillation of u_m on C_{ρ_m} can be made arbitrarily small, uniformly in $m \in \mathbb{N}$.

First note that by Fubini's theorem, if we denote arc length on C_ρ by s, from the estimate

$$\infty > c \geq \int_\Omega |\nabla u_m|^2 \, dz \geq \int_0^1 \int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds \, d\rho$$

we obtain that

$$\int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds < \infty$$

for almost every $\rho < 1$ and all $m \in \mathbb{N}$.
Choosing $\rho_0 < 1$ we may refine this estimate as follows:

$$\begin{aligned} \int_\Omega |\nabla u_m|^2 \, dz &\geq \int_{\rho_0^2}^{\rho_0} \left(\rho \int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds \right) \frac{d\rho}{\rho} \\ &\geq |\log \ \rho_0| \operatorname*{ess\,inf}_{\rho_0^2 \leq \rho \leq \rho_0} \left(\rho \int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds \right) . \end{aligned}$$

Suppose $\rho_m \in [\rho_0^2, \rho_0]$ is such that

$$\rho_m \int_{C_{\rho_m}} |\frac{\partial}{\partial s} u_m|^2 \, ds \leq 2 \operatorname*{ess\,inf}_{\rho_0^2 \leq \rho \leq \rho_0} \left(\rho \int_{C_\rho} |\frac{\partial}{\partial s} u_m|^2 \, ds \right)$$

and denote

$$C = \sup_{m \in \mathbb{N}} \int_\Omega |\nabla u_m|^2 \, dz < \infty \ .$$

Fix $z_j = e^{i\phi_j}$, $j = 1, 2$, the points of intersection of $C_{\rho_0^2}$ with $\partial\Omega$, $\phi_1 < \phi_0 < \phi_2$. Also denote $z_j^m = e^{i\phi_j^m}$, $j = 1, 2$, with $\phi_1^m < \phi_0 < \phi_2^m$ the points of intersection of C_{ρ_m} with $\partial\Omega$.

Then $\phi_1^m \in I_1$, $\phi_2^m \in I_2$ while by Hölder's inequality

$$\begin{aligned} |u_m(z_1^m) - u_m(z_2^m)|^2 &\leq \left(\int_{C_{\rho_m}} |\frac{\partial}{\partial s} u_m| \, ds \right)^2 \\ &\leq \pi \rho_m \int_{C_{\rho_m}} |\frac{\partial}{\partial s} u_m|^2 \, ds \leq \frac{2\pi C}{|\log \ \rho_0|} < \varepsilon \end{aligned} \tag{2.14}$$

if $\rho_0 > 0$ is sufficiently small. This estimate being uniform in m, for large $m \geq m_0(\phi_1, \phi_2)$ we obtain a contradiction to (2.13) and the proof is complete.
□

Remark. From (2.14), monotonicity (2.10), the assumption that Γ is a Jordan curve – that is, a homeomorphic image of the circle S^1 – and the three-point condition, also a direct proof of equi-continuity of the sub-level sets of E in $C^*(\Gamma)$ can be given; see Courant [1; Lemma 3.2, p. 103] or Struwe [17; Lemma I.4.3]. Moreover, note that (2.14) implies a uniform estimate for the modulus of continuity of a function $u \in H_0^{1,2}(\Omega)$ on a sequence of concentric circular arcs around any fixed center $z_0 \in \overline{\Omega}$ in terms of its Dirichlet integral. This observation was used by Lebesgue [1] to obtain an equi-continuous minimizing sequence for Dirichlet's integral in his solution of the classical Dirichlet problem; see also Section 5.7.

Proof of Theorem 2.8. By (2.10) and the generalized Poincaré inequality Theorem A.9 of Appendix A, for $u \in C^*(\Gamma)$ there holds

$$\int_\Omega |u|^2\,dz \le c\left(\int_\Omega |\nabla u|^2\,dz + \int_{\partial\Omega} |u|^2\,do\right) \le cE(u) + c(\Gamma).$$

Thus E is coercive on $M = C^*(\Gamma)$ in $H^{1,2}(\Omega;\mathbb{R}^3)$. Moreover E is weakly lower semi-continuous on $H^{1,2}(\Omega;\mathbb{R}^3)$. By Theorem 1.2 and Lemma 2.9 therefore the functional E achieves its infimum in $C^*(\Gamma)$, which by conformal invariance equals that in $C(\Gamma)$. □

2.10 Regularity. As in the preceding examples we may ask what regularity properties the minimizer u and the parametrized surface possess. We note a few results:
(1°) $u\big|_{\partial\Omega}$ is strictly monotone (Douglas [1], Radó [1]).
(2°) If $\Gamma \in C^{m,\alpha}$, $m \ge 1$, $0 < \alpha < 1$, then also $u \in C^{m,\alpha}(\overline{\Omega},\mathbb{R}^3)$; this result is due to Hildebrandt [1] (for $m \ge 4$) with later improvements by Nitsche [1] .
While remarks (1°), (2°) apply to arbitrary solutions of the Plateau problem (2.10)–(2.12), minimality is crucial for the next observations concerning the geometric regularity of the parametrized solution surface.
(3°) A minimizer $u \in C^*(\Gamma)$ parametrizes an immersed minimal surface $S = u(\Omega) \subset \mathbb{R}^3$; see Osserman [2], Gulliver [1], Alt [1], Gulliver-Osserman-Royden [1]; if Γ is anlytic, S is immersed up to the boundary; see Gulliver-Lesley [1]. If Γ is extreme, that is $\Gamma \subset \partial K$ where $K \subset \mathbb{R}^3$ is convex, S is embedded; see Meeks-Yau [1]. Existence of embedded minimal surfaces bounded by extreme curves independently was obtained by Tomi-Tromba [1] and Almgren-Simon [1].

2.11 Note. In the history of the calculus of variations it seems that Plateau's problem has played a very prominent role. Important developments in the general stream of ideas often were prompted by insights gained from the study of minimal surfaces. As an example, consider the classical mountain pass lemma (see also Chapter II.1) which was used by Courant [1; Chapter VI.6–7] to establish the existence of unstable minimal surfaces, previously obtained by

Morse-Tompkins [1] and Shiffman [1] by a less direct, topological reasoning. However, since this material has been covered very extensively elsewhere (see Struwe [18]), here we will confine ourselves to the above remarks.

For an introduction to Plateau's problem and minimal surfaces, see for instance, Osserman [1], or consult the encyclopedic book by Nitsche [2].

A truely remarkable – popular and profound – book on the subject is available by Hildebrandt-Tromba [1].

3. Compensated Compactness

As noted in Remark 1.8, it is conceivable that in the vector-valued case lower semi-continuity results may hold true under a weaker convexity assumption than in Theorem 1.6, provided suitable structure conditions are satisfied by the functional in variation. Weakening the convexity hypothesis is necessary, for instance, in dealing with problems arising in 3-dimensional elasticity, where we encounter energy functionals $\int_\Omega W(\nabla u)\,dx$ with a stored energy function W depending on the determinant, the minors and the eigenvalues of the deformation gradient ∇u. Since infinite volume distortion for elastic materials will afford an infinite amount of energy, it is natural to suppose that $W \to \infty$ if either $\det(\nabla u) \to 0$ or $\det(\nabla u) \to \infty$; hence W cannot be convex in ∇u. However, there is a large class of materials that can be described by *polyconvex* stored energy functions, which are of the form

$$W(\nabla u) = f(\text{subdeterminants of } \nabla u),$$

where f is convex in each of its variables. John Ball [1] was the first to see that lower semi-continuity results will hold for such functionals. The difficulty, of course, lies in proving, for instance, weak convergence $\det(\nabla u_m) \rightharpoonup \det(\nabla u)$ for a sequence $u_m \rightharpoonup u$ weakly in $H^{1,3}(\Omega, \mathbb{R}^3)$. Questions of this type had been investigated by Reshetnyak [1], [2]. A general frame for studying such problems is provided by the compensated compactness scheme of Murat and Tartar.

The basic principle of the compensated compactness method is given in the following lemma; see Tartar [2; p. 270 f.].

3.1 The compensated compactness lemma. *Let Ω be a domain in $\mathbb{R}^n$ and suppose that*

(1°) $u_m = \left(u_m^1, \dots, u_m^N\right) \rightharpoonup u$ weakly in $L^2(\Omega; \mathbb{R}^N)$.

(2°) The set $\left\{\sum_{j,k} a_{jk} \frac{\partial u_m^j}{\partial x_k} \; ; \; m \in \mathbb{N}\right\}$ is relatively compact in $H^{-1}_{loc}(\Omega; \mathbb{R}^L)$ for a set of vectors $a_{jk} \in \mathbb{R}^L$; $1 \le j \le N$, $1 \le k \le n$. Let

$$\Lambda = \left\{\lambda \in \mathbb{R}^N \; ; \; \sum_{j,k} a_{jk} \lambda_j \xi_k = 0 \text{ for some } \xi \in \mathbb{R}^n \setminus \{0\}\right\}$$