Pseudodifferential equations in polygonal domains: Regularity and numerical approximation

Heiko Gimperlein[1,2]

joint with N. Louca[1], R. Mazzeo[3], E. P. Stephan[4], J. Stocek[1]

1: Heriot-Watt University, Edinburgh
2: University of Paderborn
3: Stanford University
4: Leibniz University Hannover

Fast Boundary Element Methods in Industrial Applications
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Set up

Γ Riemannian manifold, $\Omega \subseteq \Gamma$ piecewise smooth domain
$A : H^s(\Gamma) \to H^{-s}(\Gamma)$ elliptic pseudodifferential operator, order $2s$

$$\begin{aligned} Au &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Gamma \setminus \overline{\Omega}. \end{aligned}$$

Examples:

- $A =$ weakly singular or hypersingular integral operator
- $A = (-\Delta)^s$ fractional Laplacian, $s \in (0, 1)$

Goals for this talk:

- **Fractional problems interesting** & closely related to BEM!
- **Asymptotic expansion** of solutions near edges and corners
- **Optimal approximation** on graded meshes, towards *hp*
- Extension to **transmission problems**

Integral fractional Laplacian

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u(x)), \qquad s \in (0,1)$$

- Note: generator of Lévy process and has physical meaning. Used in probability, PDE, applications.
- $W = \frac{1}{2}(-\Delta)^{1/2}$, $V = \frac{1}{2}(-\Delta)^{-1/2}$ for flat screen
- Recent numerical analysis of $(-\Delta)^s$: Ainsworth, Glusa (2017), Nochetto et al. (2017)
- Recent analysis: Caffarelli, Silvestre, Figalli ... (since 2007)
- Recent modeling: Du (continuum mechanics, ICM 2018), Perthame (cell movement, 2018), Mouhot (kinetic eqns), ...
- Don't confuse with spectral $(-\Delta)^s$, the fractional power of the Dirichlet problem (Nochetto, Otarola, ...).

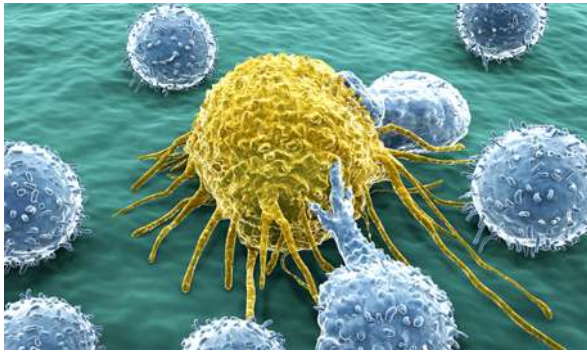
Motivation

$$(-\Delta)^s u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \Gamma \setminus \overline{\Omega}.$$

Classical Applications:

Finance / option pricing: Tankov (2003), Wilmott (1993)

Continuum Mechanics: Du (2018)



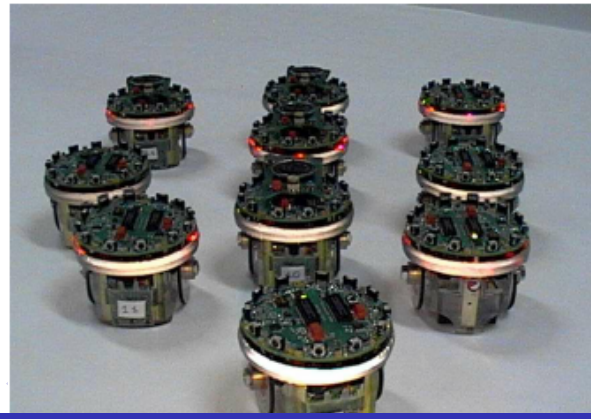
Nonlocal movement of cells and organisms

Perthame, Sun, Tang, ZAMP 2018

Estrada-Rodriguez, HG, Painter, SIAP 2018

Estrada-Rodriguez, HG, Painter, Stocek, M3AS 2019

Estrada, Estrada-Rodriguez, HG, SIAM Review 2019



Swarm robotic systems

Estrada-Rodriguez, HG, arXiv:1807.10124

Duncan, Dragone, Estrada-Rodriguez, HG, Stocek, Vargas, 2019

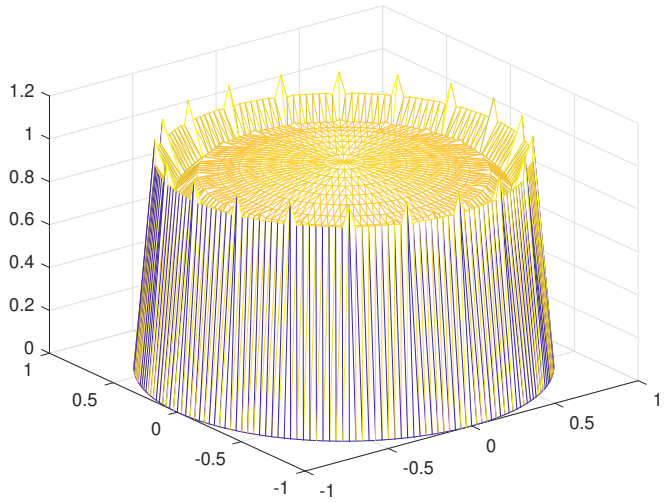
Numerical approximation: non-trivial

$$(-\Delta)^s u = 1 \text{ in } \Omega$$
$$u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}.$$

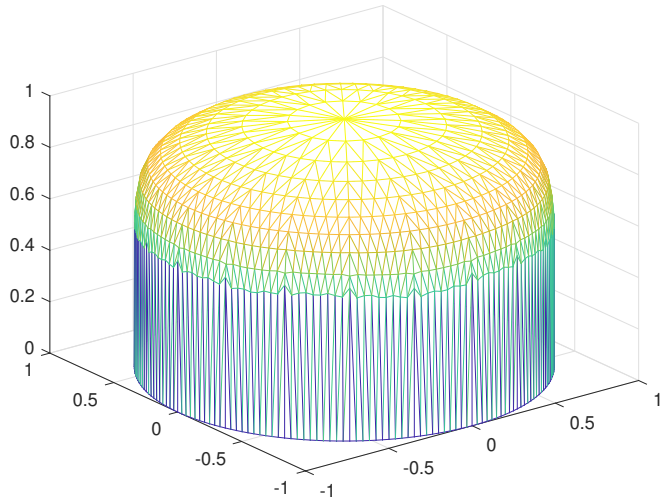
For $\Omega = \{|x| < 1\}$. $s = \frac{1}{10}$

Exact solution: $u(x) = (1 - |x|^2)_+^{\frac{1}{10}}$

Uniform mesh



2-graded mesh



Numerical Results

$$
\begin{aligned}
(-\Delta)^s u &= 1 \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}.
\end{aligned}
$$

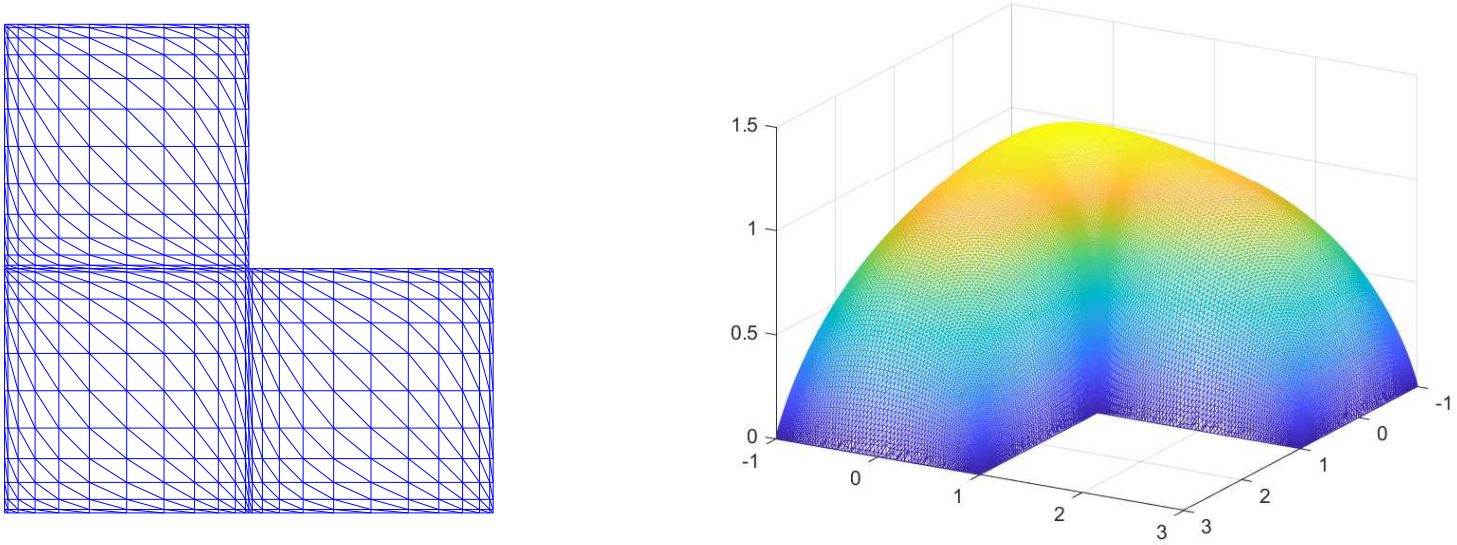
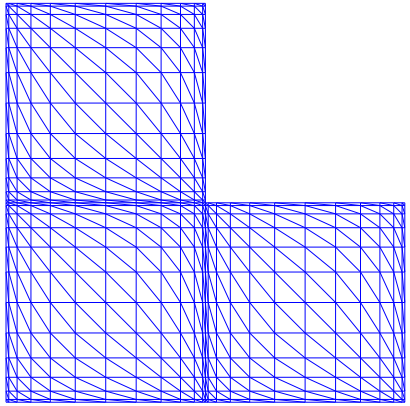


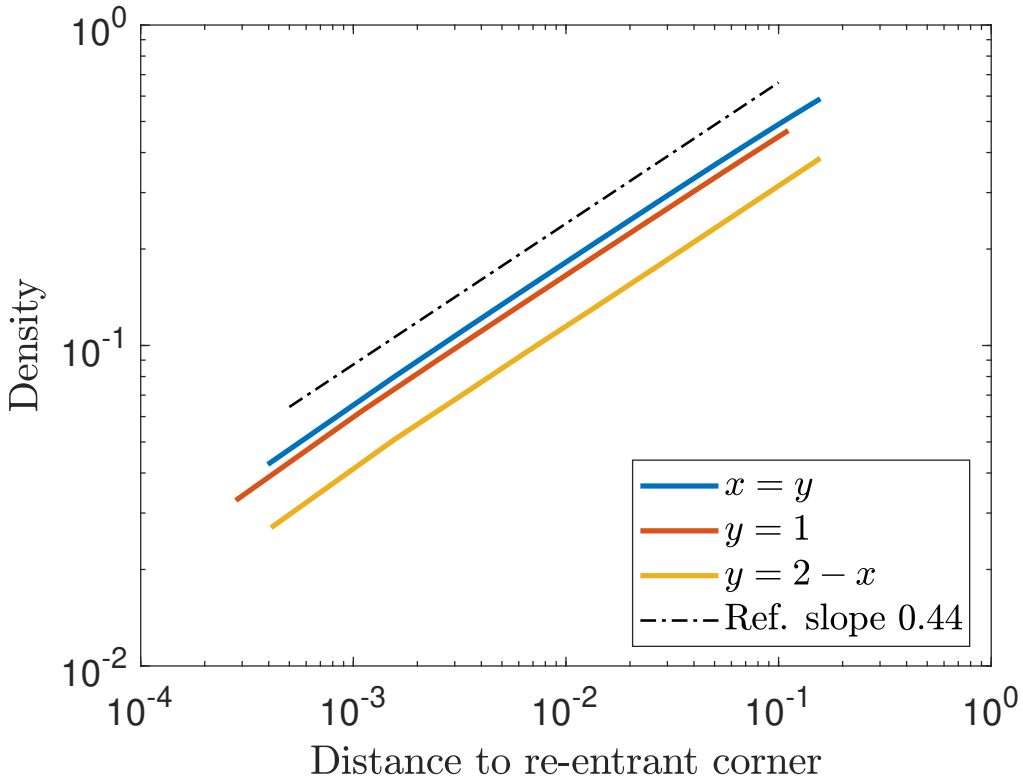
Figure: Algebraically 2–graded mesh for L–shape (left) and numerical solution with $s = \frac{1}{2}$ (right).

Edge and corner singularities

$$\begin{aligned}(-\Delta)^s u &= 1 \quad \text{in } \Omega,\\ u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}.\end{aligned} \qquad (\star)$$

Numerically $u(x) \sim \mathrm{dist}(x, \mathrm{corner})^{\nu_c}$.





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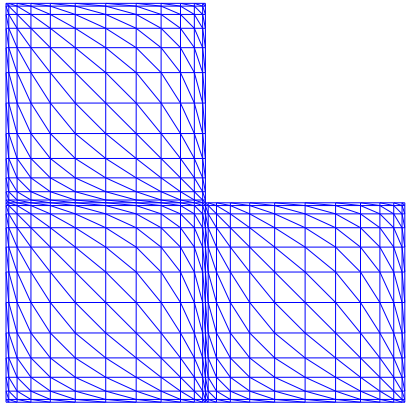
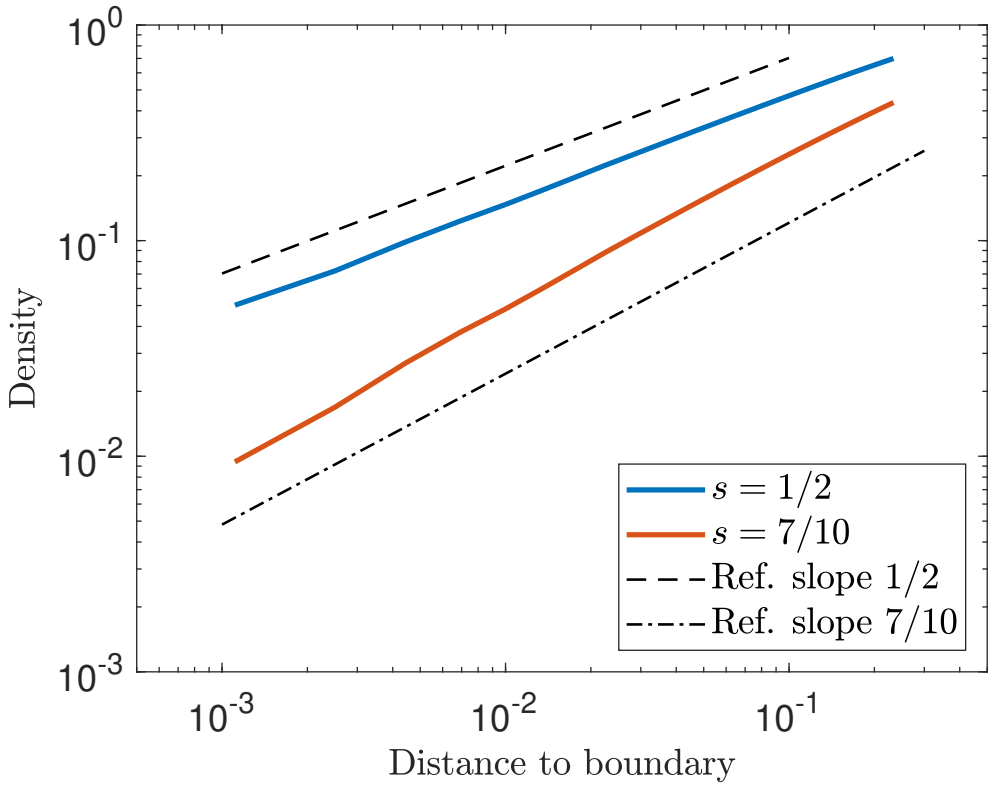




Edge and corner singularities

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

This talk concerns a singular analysis approach to:

Theorem

For Ω polygon and $f \in C^\infty(\overline{\Omega})$, the solution u to $(\star)$ admits an asymptotic expansion at the edges and corners:
Edge E: $u(x) \sim \mathrm{dist}(x, E)^s$,
Corner C: $u(x) \sim \mathrm{dist}(x, C)^{\nu_c}$.
Here, ν_c relates to the smallest eigenvalue of an elliptic 2nd order differential operator on S^2_+.

Previous work: Edge behaviour of solutions

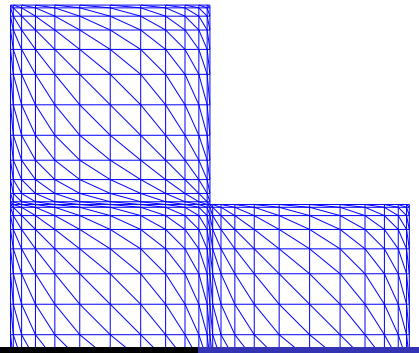
Theorem (Ros-Oton, Serra 2017 / Grubb 2015)

Let $s \neq \frac{1}{2}$, Ω *sufficiently smooth,* $f \in L^\infty(\Omega)$.
Then $\frac{u(x)}{\mathrm{dist}(x,\partial\Omega)^s} \in C^\alpha$ *for some* $\alpha > 0$.
Logarithmic corrections for $s = \frac{1}{2}$.

Theorem (Acosta 2017)

Let Ω *convex polyhedron. Then quasi-optimal convergence on* $\beta-$*graded meshes:*

$$\|u - u_h\|_{H^s} \lesssim h^{\min\{\frac{\beta}{2}, 2-s\}-\varepsilon}.$$



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We provide a new approach to these results, based on an analysis of a degenerate elliptic PDE
$\longrightarrow$ singular expansions in polyhedral domains.

Setup and existence of solutions

Sobolev Spaces:

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\mathcal{F}u(|\xi)|^2 \, d\xi < \infty \right\}$$

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dy \, dx$$

Fractional Dirichlet Problem:

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned}$$

Variational Formulation: Find $u \in \widetilde{H}^s(\Omega)$ s.t. for all $v \in \widetilde{H}^s(\Omega)$

$$\frac{c_{n,s}}{2} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dy \, dx = \int_\Omega f(x) v(x) \, dx$$

Lax-Milgram $\rightsquigarrow$ $\exists!$ solution in $\widetilde{H}^s(\Omega)$.
Higher regularity of solutions?

Extension approach, $s = 1/2$

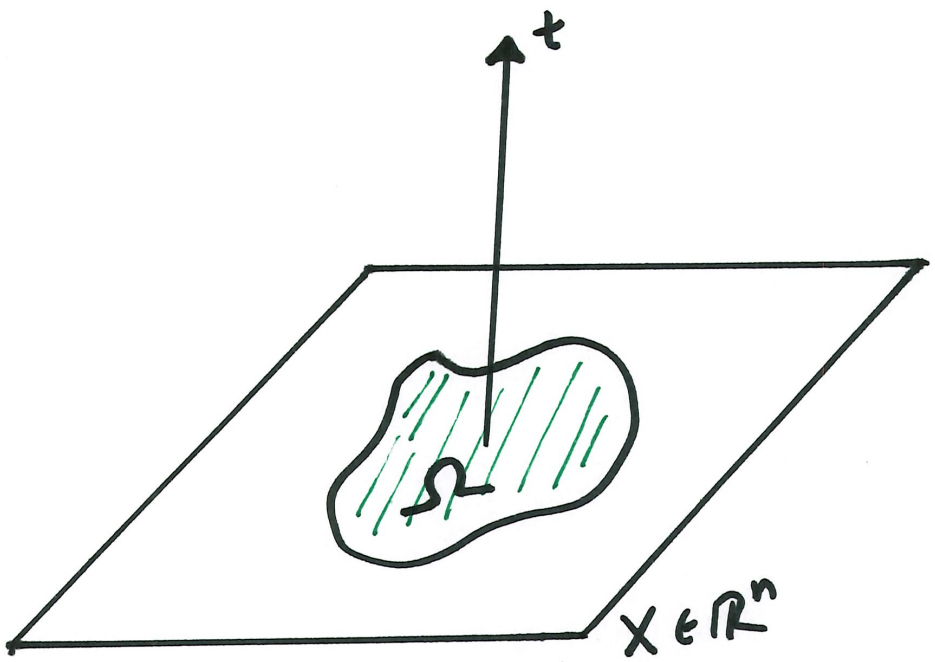
Fractional Dirichlet problem:

$$\begin{aligned} (-\Delta)^{\frac{1}{2}} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \tag{$\star$}$$

Harmonic extension of u:

$$\begin{aligned} \Delta_{(X,t)} U(X,t) &= 0 && \text{for } X \in \Omega \text{ and } t > 0 \\ U(X,0) &= u(X) && \text{for } X \in \mathbb{R}^n \end{aligned}$$

Extension approach, $s = 1/2$



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Mixed problem:

$$\begin{aligned} \Delta_{(X,t)} U(X,t) &= 0 && \text{for } X \in \Omega \text{ and } t > 0 \\ -\lim_{t \to 0^+} \partial_t U(X,t) &= f(X) && \text{for } X \in \Omega \\ U(X,0) &= 0 && \text{for } X \in \mathbb{R}^n \setminus \overline{\Omega} \end{aligned} \tag{$\star\star$}$$

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Theorem (Caffarelli, Silvestre 2007, $\Omega = \mathbb{R}^n$)

- $(-\Delta)^{1/2}$ *coincides with the Dirichlet-to-Neumann operator:*

$$T : u \mapsto -U_t(x,0).$$

- $(\star) \Leftrightarrow (\star\star) \Leftrightarrow Tu = f$.

Extension approach, $s \neq 1/2$

Fractional Dirichlet problem:

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Extension: (degenerately elliptic for $s \neq 1/2$)

$$\begin{aligned} \nabla_{(X,t)} \cdot (t^{1-2s} \nabla_{(X,t)} U(X,t)) &= 0 && \text{for } X \in \mathbb{R}^n \text{ and } t > 0 \\ U(X,0) &= u(X) && \text{for } X \in \mathbb{R}^n \end{aligned} .$$

Extension approach, $s \neq 1/2$

Fractional Dirichlet problem:

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Mixed PDE problem:

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Theorem (Caffarelli, Silvestre 2007, $\Omega = \mathbb{R}^n$)

- $(-\Delta)^s u = Tu = c_{n,s} \lim_{t \to 0^+} -t^{1-2s} \partial_t U.$
- $(\star) \Leftrightarrow (\star\star\star) \Leftrightarrow Tu = f.$

Extension approach to fractional boundary problems

Want to solve

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Again, use operator in one dimension higher

$$L_s U(X,t) := t^{2s-1}\nabla_{X,t}\cdot\big(t^{1-2s}\nabla_{X,t}U(X,t)\big) = \partial_t^2 U + \tfrac{1-2s}{t}\partial_t U + \Delta_X U.$$

Mixed boundary problem

$$\begin{aligned} L_s U(X,t) &= 0 \quad \text{for } X \in \Omega,\ t > 0, \\ U(X,0) &= 0 \quad \text{for } X \in \mathbb{R}^n \setminus \overline{\Omega}, \\ -\lim_{t\to 0^+} t^{1-2s}\partial_t U(X,t) &= f(X) \quad \text{for } X \in \Omega,\ t > 0. \end{aligned} \qquad \text{(MBP)}$$

Key observation: $(\star) \Leftrightarrow$ (MBP) also for $\Omega \subset \mathbb{R}^n$.

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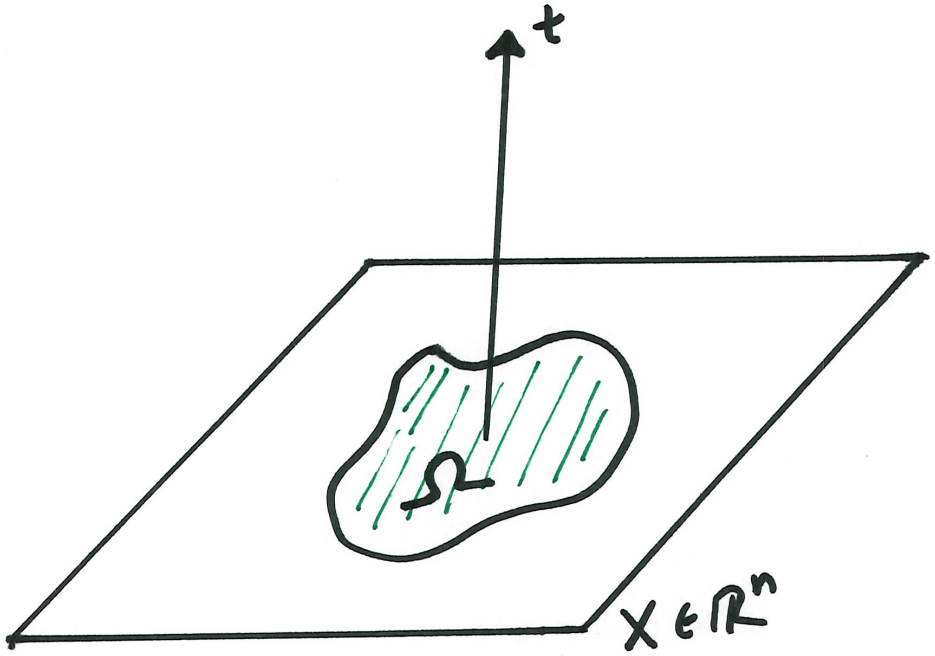
Key observation: $(\star) \Leftrightarrow$ (MBP) also for $\Omega \subset \mathbb{R}^n$.

Observation

on flat screen: $W = \frac{1}{2}(-\Delta)^{1/2}$, $V = \frac{1}{2}(-\Delta)^{-1/2}$

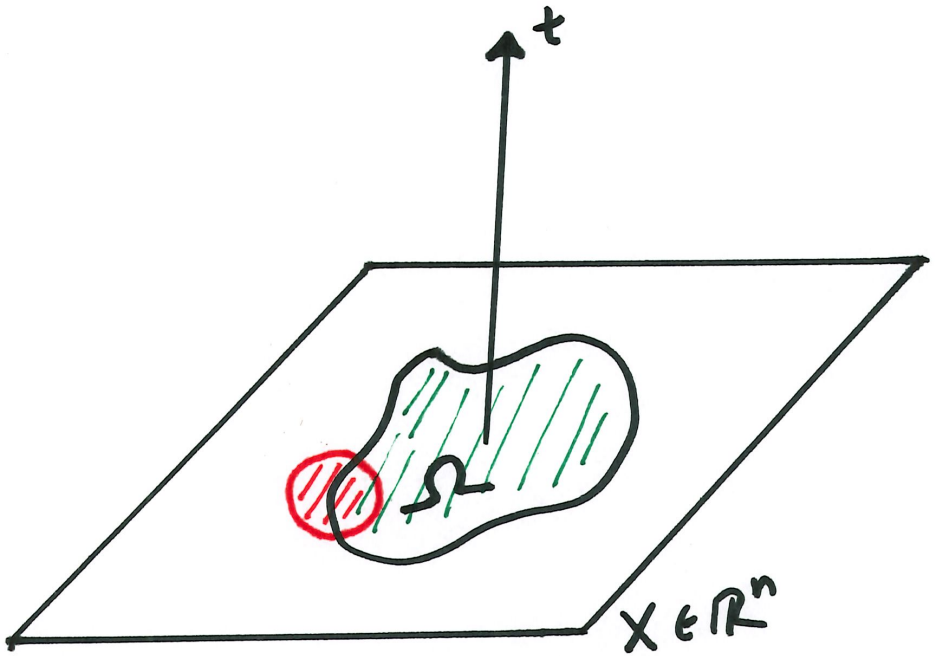
Extension approach to fractional boundary problems

$$\begin{aligned}(-\Delta)^s u &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}.\end{aligned}$$



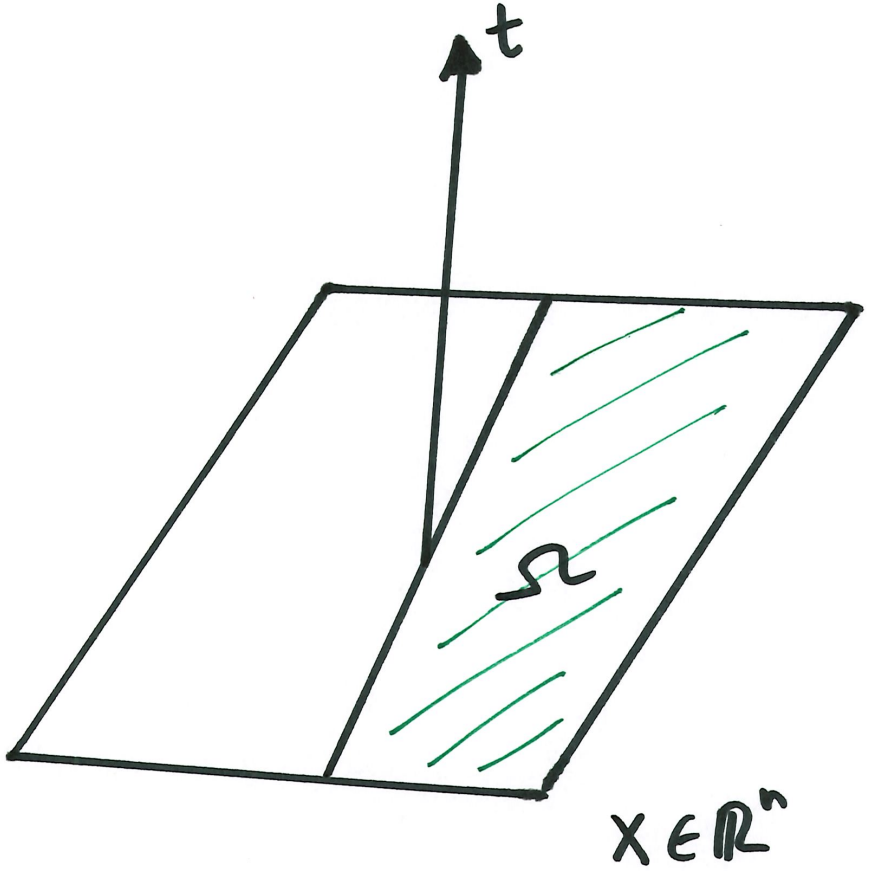
Extension approach to fractional boundary problems

Localisation:



Extension approach to fractional boundary problems

$$\begin{aligned} L_s U(X,t) &= 0 && \text{for } X \in \Omega,\ t > 0, \\ U(X,0) &= 0 && \text{for } X \in \mathbb{R}^n \setminus \overline{\Omega}, \\ \lim_{t\to 0^+} t^{1-2s}\partial_t U(X,t) &= f(X) && \text{for } X \in \Omega,\ t > 0. \end{aligned}$$



Extension approach to fractional boundary problems

Coordinates in half-space $(X,t) \in \mathbb{R}^{n+1}_+$:
$X = (x,y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $t > 0$.
Fourier Transform:

$$\mathcal{F}_{y\to\eta} L_s = \partial_t^2 + \frac{1-2s}{t}\partial_t + \partial_x^2 - |\eta|^2 =: \widehat{L_s}$$

Mixed boundary problem now becomes

$$\begin{aligned} \widehat{L_s}\widehat{U}(x,\eta,t) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ \widehat{U}(x,\eta,0) &= 0 \quad \text{for } x<0, \\ \lim_{t\to 0^+} t^{1-2s}\partial_t \widehat{U}(x,\eta,0) &= \widehat{u}(x,\eta,0) \quad \text{for } x>0. \end{aligned} \qquad (\widehat{\text{MBP}})$$

Local behaviour near $\partial\Omega$: $|\eta|^2$ lower order $\to$ Consider $\eta = 0$ first.
Polar coordinates:

$$t = \rho\sin\theta,\ x = \rho\cos\theta$$

$$B(L_s) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right).$$

Extension approach to fractional boundary problems

$$B\left(L_s\right) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\underbrace{\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right)}_{:=P_s}$$

Asymptotics from separation of variables:
$U = \rho^\nu \phi(\theta) \to$ eigenvalue problem:

$$P_s\phi = -\lambda(\nu)\phi$$

with the mixed boundary conditions

$$\theta^{1-2s}\partial_\theta\phi = 0 \quad \text{at } \theta = 0, \qquad \phi = 0 \qquad \text{at } \theta = \pi$$

where $\lambda(\nu) = (\nu + \frac{1}{2})^2 - (s - \frac{1}{2})^2$.

Extension approach to fractional boundary problems

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Lemma

P_s is self-adjoint and positive on $L^2\left([0,\pi], \sin^{1-2s}(\theta)\, d\theta\right)$

Extension approach to fractional boundary problems

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Lemma

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Classical (Zaremba) $s = 1/2$:
Eigenvalues: $\lambda_j = \left(j + \frac{1}{2}\right)^2$, eigenfunctions: $\phi_j = \cos\left(\left(j + \frac{1}{2}\right)\theta\right)$,
Singular expansion $u \sim \sum u_j(y) x^{j+\frac{1}{2}}$ with exponents $\nu_j = j + \frac{1}{2}$.

Extension approach to fractional boundary problems

$$B\left(L_s\right) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\underbrace{\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right)}_{:=P_s}$$

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Lemma

P_s is self-adjoint and positive on $L^2\left([0,\pi], \sin^{1-2s}(\theta)\, d\theta\right)$
$\lambda_j^s = (j+s)(j+1-s)$, $\varphi_j^s = \sin^s(\theta)P_j^s(\cos(\theta))$,
Singular exponents: $\nu_j^s = j + s$.
Here P_j^s is the associated Legendre function of the first kind.

Extension approach to fractional boundary problems

$$B\left(L_s\right) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\underbrace{\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right)}_{:=P_s}$$

Asymptotics from separation of variables:

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where $\lambda(\nu) = (\nu + \frac{1}{2})^2 - (s - \frac{1}{2})^2$.

Corollary

The solution to $(-\Delta)^s u = f$ admits a polyhomogeneous expansion near $\partial\Omega$ with exponents $\nu_j = j + s$, i.e. $u \sim \sum u_j(y)x^{j+s}$.

Recovers Grubb 2015; approach generalizes to manifolds with corners, e.g. polygons $\subseteq \mathbb{R}^2$, and lower-order perturbations.

Structure of solution operator

The analysis of the model problem in the half space implies corresponding results for smooth $\Omega \subseteq \Gamma$, using pseudodifferential techniques.

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Solution operator $\mathcal{S} : f \mapsto u$ has distributional kernel $K(X, Y)$:

$$u(X) = \mathcal{S}f(X) = \int K(X, Y) f(Y)\, dY.$$

Theorem (HG, Louca, Mazzeo)

- *Distributional kernel $K(X, Y)$ is C^∞ in $\Omega \times \Omega \setminus \triangle\Omega$.*
- *$K(X, Y)$ is a conormal distribution with asymptotic expansions at $\triangle\Omega$, $\Omega \times \partial\Omega$, $\partial\Omega \times \Omega$, $\partial\Omega \times \partial\Omega$ (in suitable "polar coordinates").*
- *$\mathcal{S}$ belongs to a large edge pseudodifferential calculus of Mazzeo '91 / Mazzeo-Witten '17.*

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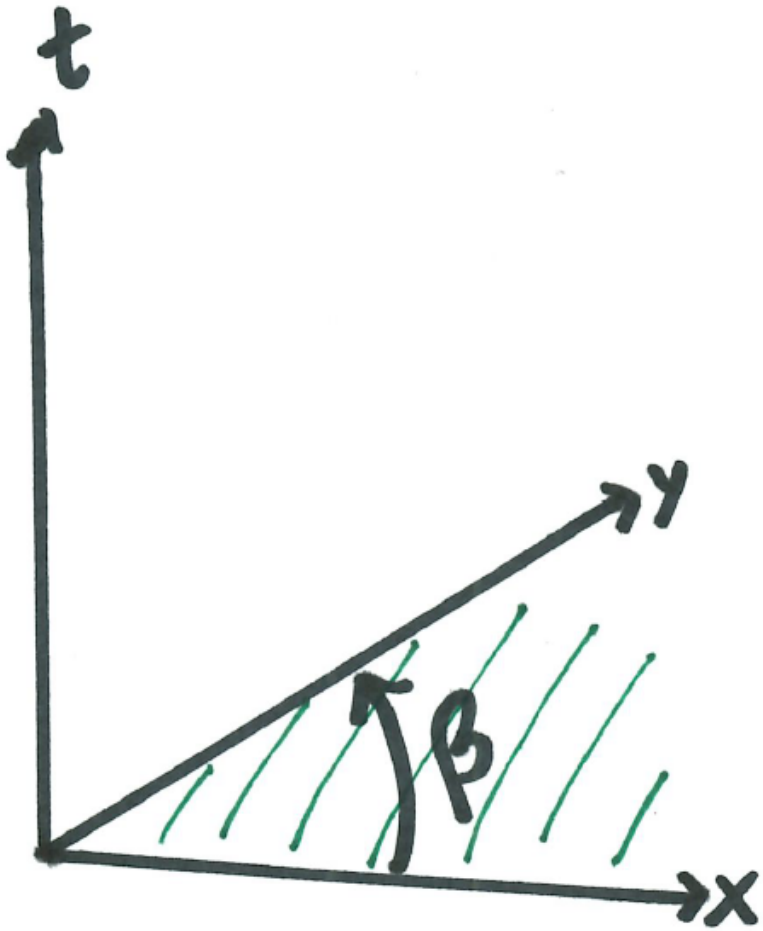
The proof of this theorem proceeds with a corresponding fine analysis of the solution operator to the extended problem for L_s in $\mathbb{R}^{n+1}_+$ with mixed boundary conditions.

Polygonal domains: Model problem

Corner problem: opening angle β



Extension approach for polygons

Spherical coordinates

$$t = r\sin(\theta),\ \ x = r\cos(\theta)\cos(\phi),\ \ y = r\cos(\theta)\sin(\phi)$$

$$\widetilde{L}_s U(r,\theta,\varphi) = \partial_r^2 U + \frac{3-2s}{r}\partial_r U + \frac{1}{r^2}\underbrace{\left(\Delta_{\theta,\varphi} U + (1-2s)\tan(\theta)\partial_\theta U\right)}_{=:\widetilde{P}_s U}.$$

Here $\Delta_{\theta,\varphi}$ is the Laplace-Beltrami operator on the 2-sphere S^2:

$$\Delta_{\theta,\varphi} U = \frac{1}{\sin(\theta)}\partial_\theta(\sin(\theta)\partial_\theta U) + \frac{1}{\sin^2(\theta)}\partial_\varphi^2 U.$$

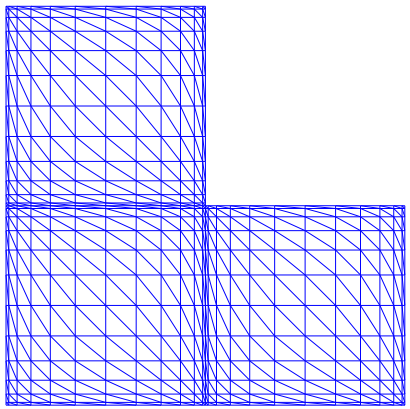
Separation of variables: $\rightsquigarrow$ self-adjoint spectral problem:

$$\begin{aligned} \widetilde{P}_s\psi &= -\lambda\psi \quad \text{on } S^2_+, \\ \theta^{1-2s}\lim_{\theta\to 0^+}\partial_\theta\psi &= 0 \quad \text{for } \phi\in(0,\beta), \\ \psi &= 0 \quad \text{for } \phi\notin(0,\beta), \theta=\pi/2. \end{aligned}$$

Asymptotic expansion near corner: $u \sim \sum u_j r^{\nu_c(\lambda_j)}\psi_j$.

Recall: Numerical experiments for corner singularities

$$\begin{aligned} (-\Delta)^s u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$



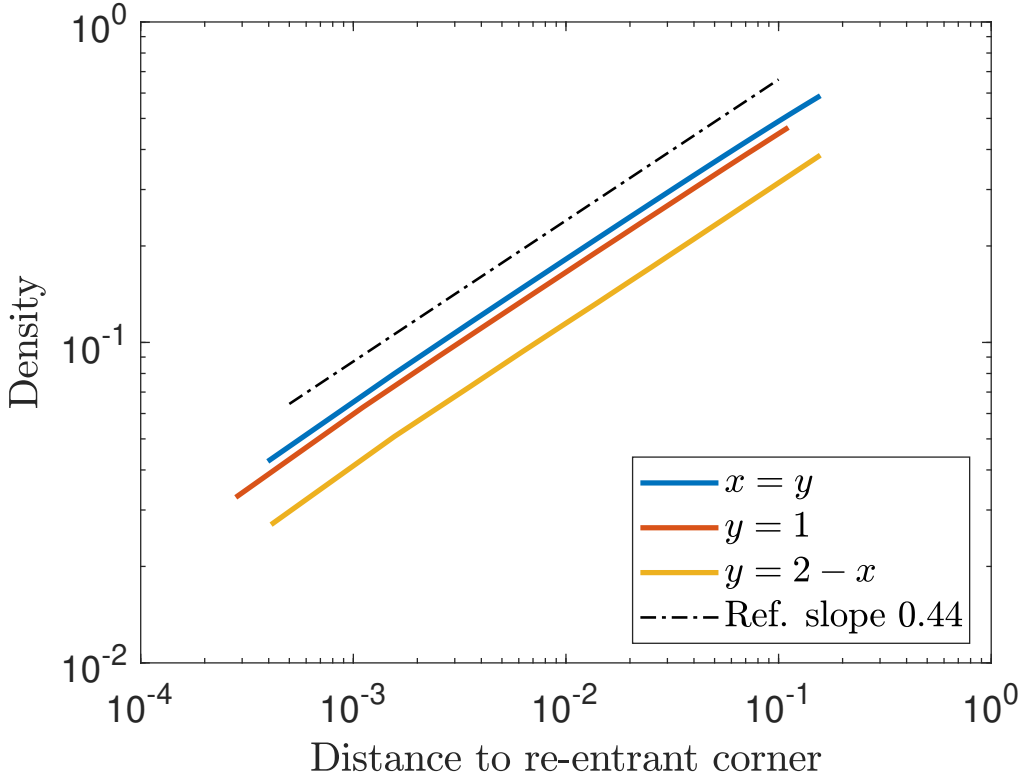


Figure: Asymptotic behaviour of u near re-entrant corner for $s = \frac{7}{10}$ (left)

Angle dependence of corner exponent, $s = \frac{1}{2}$

$s = \frac{1}{2}$ classical: J. A. Morrison, J. A. Lewis '76, Walden '74.

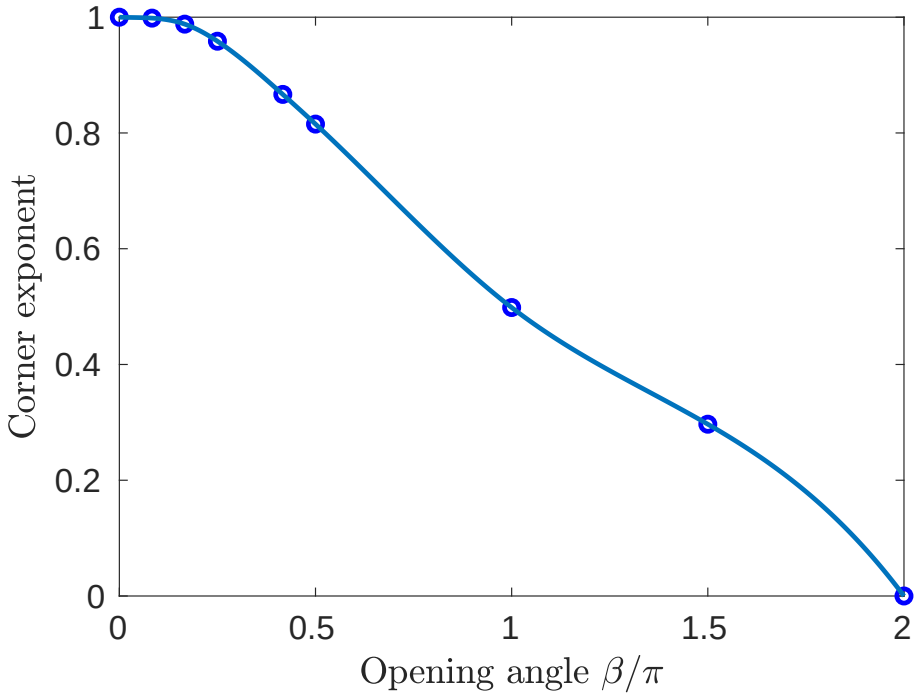


Figure: Corner exponents against opening angle, $s = \frac{1}{2}$.

Angle and s dependence of corner exponent

$s = \frac{1}{2}$ classical: J. A. Morrison, J. A. Lewis '76, Walden '74.

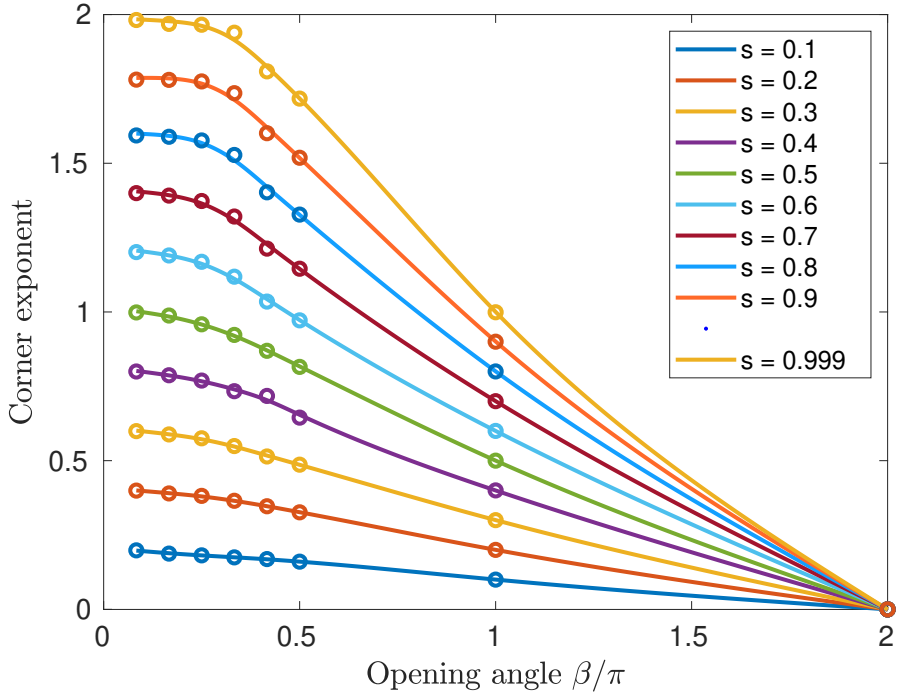


Figure: Corner exponents against opening angle for different values of s.

Convergence rates on graded meshes

Corollary (work in progress)

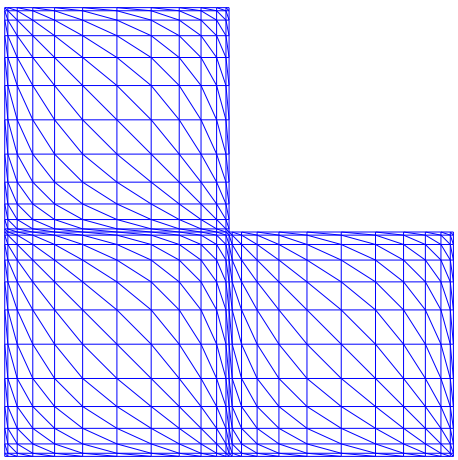
Let Ω be a polygon, $\Pi_h u$ be the best approximation to u in the $\widetilde{H}^s(\Omega)$ norm by continuous p.w. linear functions on a β-graded mesh. Then

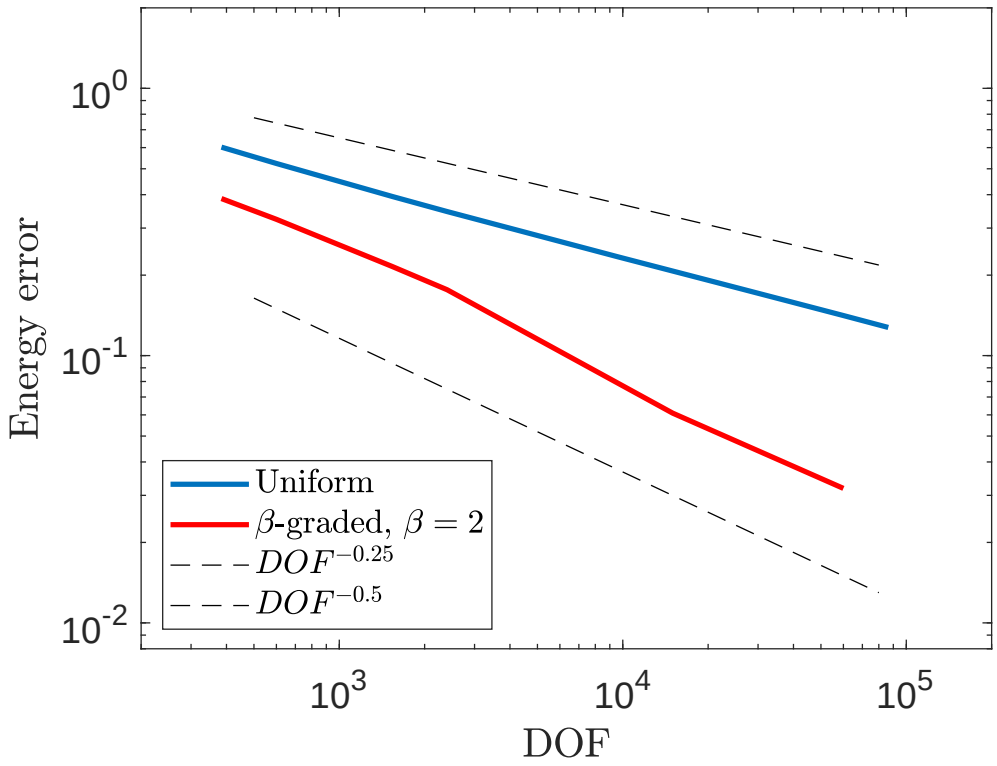
$$\|u - \Pi_h u\|_{\widetilde{H}^s(\Omega)} \lesssim h^{\min\{\frac{\beta}{2}, 2-s\}} |\log^*(h)|.$$

This generalizes results by Acosta (2017) for convex domains.
$|\log^*(h)|$ only needed for $s = \frac{1}{2}$.
Related ongoing work by J.P. Borthagaray, R. Nochetto, using different techniques.

Convergence rates on graded meshes

$$\begin{aligned} (-\Delta)^s u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$





hp convergence rates?

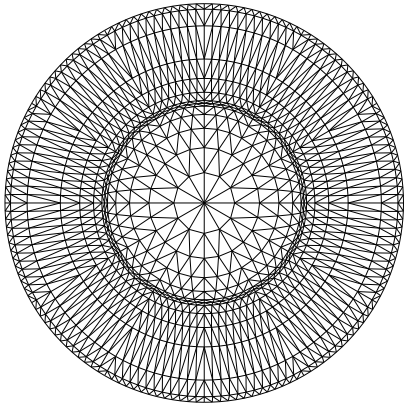
Corollary (work in progress)

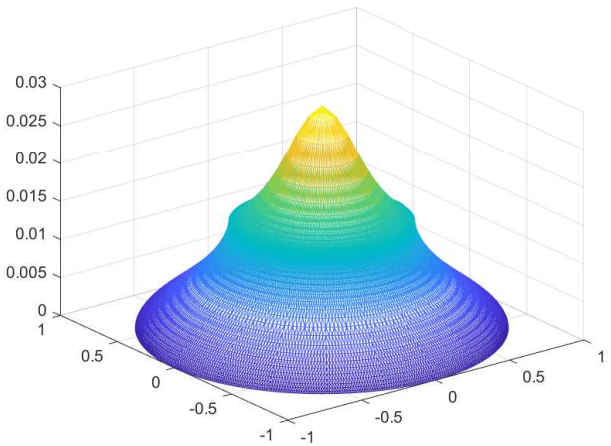
Let Ω be a polygon, $f \in C^\infty(\overline{\Omega})$,
$\Pi_{hp}u$ best approximation in the $\widetilde{H}^s(\Omega)$ norm to u by continuous p.w. polynomials of degree p on quasi-uniform mesh.

Then

$$\|u - \Pi_{hp}u\|_{\widetilde{H}^s(\Omega)} \lesssim \left(\frac{h}{p^2}\right)^{\min\{\frac{1}{2},\nu_c+1-s\}} |\log^*(h/p^2)|.$$

Singularities in interface problems





Local–nonlocal solution: $s_1 = 1,\ s_2 = \frac{7}{10},\ D_1 = D_2 = 1.$

Caffarelli, Kriventsov, etc.:

$$a(u,v) = \frac{D_1\, c_{n,s_1}}{2} \iint_{\Omega_1 \times \mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s_1}}\, dydx$$
$$+ \frac{D_2\, c_{n,s_2}}{2} \iint_{\Omega_2 \times \mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s_2}}\, dydx$$

Ideas and results

- Extension approach does not lead to physical interface problems.
- Mellin transform in direction normal to interface
- **Theorem:** Solution u admits asymptotic expansion $u \sim \sum_j u_j^1 \, (x_n)^{\nu_j^1}$ in Ω_1, $u \sim \sum_j u_j^2 \, (x_n)^{\nu_j^2}$ in Ω_2 Leading exponents $\nu^1 := \nu_0^1$, $\nu^2 := \nu_0^2$ determined by explicit system:

$$0 = \det \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}, \qquad A_{i,j} = A_{i,j}(\nu^1, \nu^2, D_1, D_2, s_1, s_2, n)$$

- **Theorem:** Optimal convergence rates on β-graded meshes.

Case 1: Different orders, $s_1 = 1$, $s_2 =: s$

Singular exponents: $2-\nu$, ν. Determined by

$$0 = \det \begin{pmatrix} D_1 \tilde{C} & -D_2 c_{n,s} C(s) \mathbf{B}(\nu+1, 2s-\nu) \\ 0 & D_2 c_{n,s} C(s) q(\nu, s) \end{pmatrix}$$

$$= D_1 \tilde{C} D_2 c_{n,s} C(s) q(\nu, s) = \frac{D_1 \tilde{C}}{\pi} \Gamma(2s-\nu)\Gamma(1+\nu)\sin(\pi(s-\nu))$$

I.e. $\nu \in s + \mathbb{N}$, independent of $D_1, D_2 > 0$.

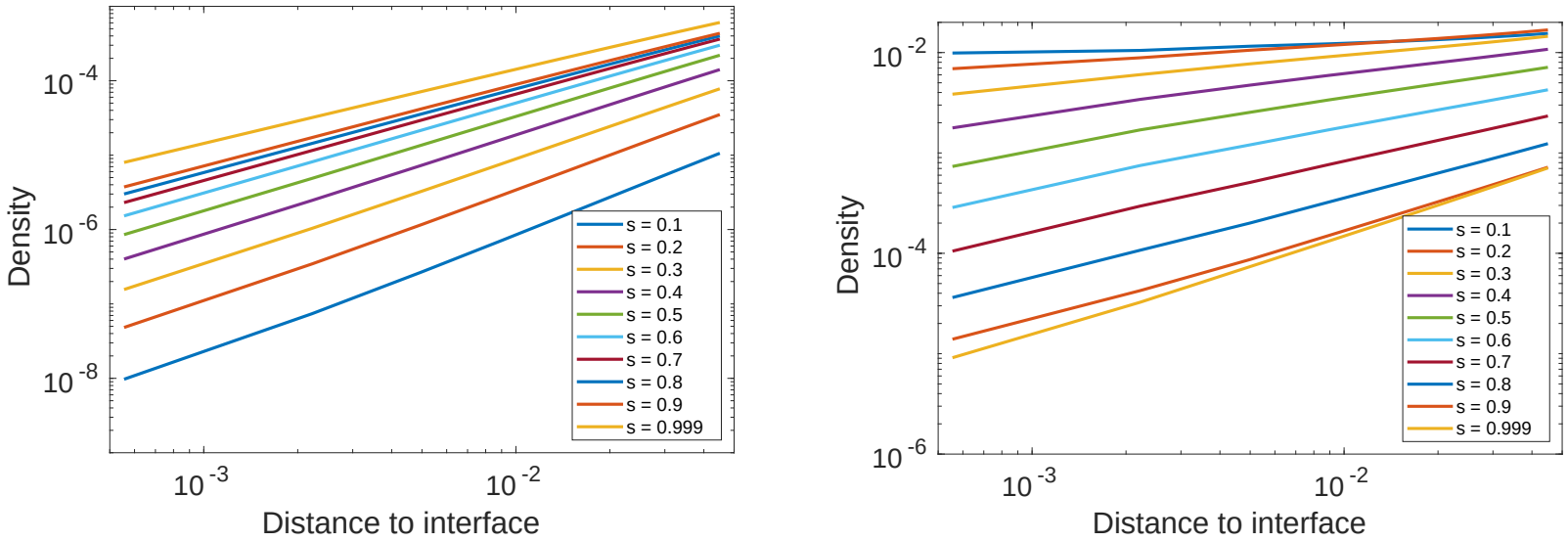


Figure: Asymptotic behavior of solution in regions 1 / 2.

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Agreement of analysis and numerical experiments:
Singular exponents in Ω^1, Ω^2

	s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Ω^1	2-s	1.93	1.83	1.74	1.62	1.51	1.40	1.30	1.20	1.11
Ω^2	s	0.10	0.20	0.29	0.39	0.49	0.59	0.69	0.80	0.91

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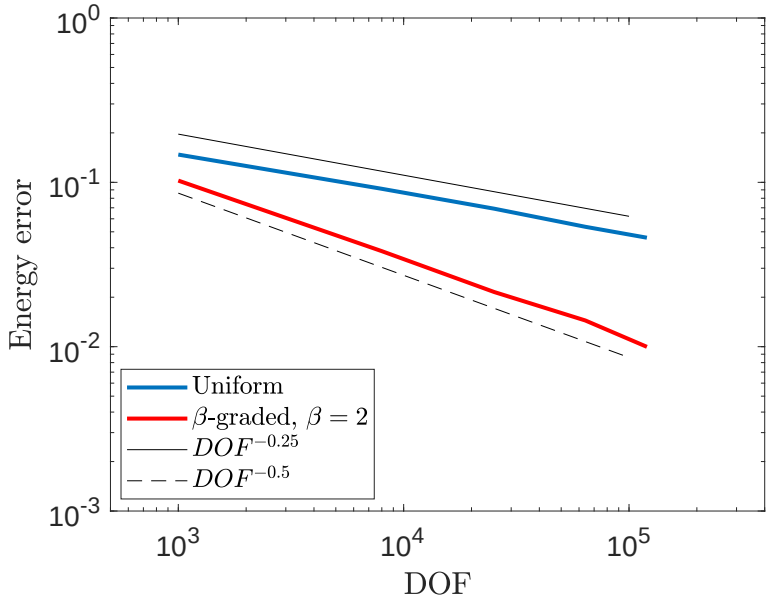
I.e. $\nu \in s + \mathbb{N}$, independent of $D_1, D_2 > 0$.

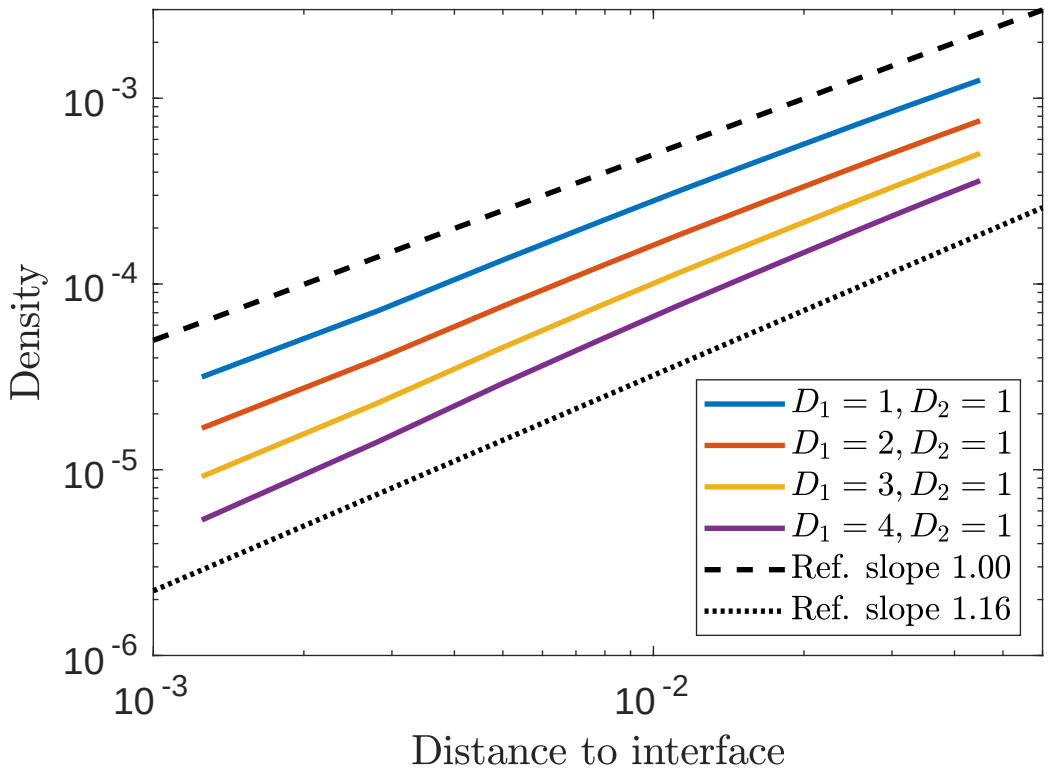
Agreement of analysis and numerical experiments:
Convergence in energy norm for $s = \frac{7}{10}$.



Case 2: Different diffusion coefficients, $s_1 = s_2 =: s$

Singular exponent ν at interface determined by

$$0 = \det \begin{pmatrix} D_1 \left(q(\nu, s) - \frac{1}{4s}\right) + D_2 \frac{1}{4s} & -\left(\frac{D_1+D_2}{2}\right) \mathbf{B}(\nu+1, 2s-\nu) \\ -\left(\frac{D_1+D_2}{2}\right) \mathbf{B}(\nu+1, 2s-\alpha) & D_2 \left(q(\nu, s) - \frac{1}{4s}\right) + D_1 \frac{1}{4s} \end{pmatrix}$$



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Agreement of analysis and numerical experiments:

	$s = 0.7$		$s = 0.8$		$s = 0.9$		$s = 0.999$	
(D_1, D_2)	Num.	Ana.	Num.	Ana.	Num.	Ana.	Num.	Ana.
$(1,1)$	1.01	1.00	1.02	1.00	1.03	1.00	0.99	1.00
$(2,1)$	1.05	1.05	1.04	1.04	1.03	1.02	1.00	1.00
$(3,1)$	1.15	1.17	1.10	1.10	1.06	1.05	1.00	1.00
$(4,1)$			1.16	1.16	1.09	1.09	1.01	1.00

Conclusion

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \tag{$\star$}$$

- Analysis using local degenerate elliptic problem in $\mathbb{R}^{n+1}_+$ with mixed boundary conditions
- Refines results of Grubb (2015), Ros-Oton–Serra (2017) for smooth Ω, extends to polygons / manifolds with corners
- Asymptotic expansions imply quasi-optimal convergence rates on graded meshes
- Using Mellin techniques, results extend to interface problems
- To do:
  mapping properties of $(-\Delta)^s$ and solution operator, hp?