Analytic globalizations of Harish-Chandra modules

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(joint work with Bernhard Krötz, Henrik Schlichtkrull)

Let G be a real reductive Lie group and $K < G$ a maximal compact subgroup. We consider representations (π, E) of G on a locally convex Hausdorff space E. A vector $v \in E$ is said to be *analytic* provided that the orbit map $\gamma_v : G \to E$, $x \mapsto \pi(x)v$, extends to a holomorphic function on a left–G–invariant neighborhood in $G_{\mathbb{C}}$. The space E^ω of analytic vectors is endowed with a natural inductive limit topology $E^\omega = \lim_{n\to\infty} E_n$,

$$E_n = \{v \in E \mid \gamma_v \text{ extends to a holomorphic map } GV_n \to E\} ,$$

indexed by a neighborhood basis $\{V_n\}_{n\in\mathbb{N}}$ of the identity in $G_{\mathbb{C}}$, and (π, E) is said to be *analytic* provided that $E = E^\omega$ as topological vector spaces. No completeness assumptions on E are imposed, so that the quotient of an analytic representation by a closed invariant subspace is again analytic.

Moderately growing analytic representations allow for an additional action by an algebra of superexponentially decaying functions. To be specific, consider a Banach representation (π, E). Fix a left–invariant Riemannian metric on G and let d be the associated distance function. The continuous functions on G decaying faster than $e^{-n\mathrm{d}(\cdot,\mathbf{1})}$ for all $n \in \mathbb{N}$ form a convolution algebra $\mathcal{R}(G)$, which is a G–module under the left regular representation. If we denote the space of analytic vectors of $\mathcal{R}(G)$ by $\mathcal{A}(G)$, the map

$$\Pi : \mathcal{A}(G) \to \mathrm{End}(E^\omega),\ \Pi(f)v = \int_G f(x)\ \pi(x)v\ dx\ , \tag{1}$$

gives rise to a continuous algebra action on E^ω. More general representations on a sequentially complete space will be called $\mathcal{A}(G)$*–tempered* provided that the integral in (1) converges strongly and defines a continuous action of $\mathcal{A}(G)$.

Recall that to an admissible G-representation (π, E) of finite length one can associate the Harish-Chandra module E_K of its K-finite vectors and that $E_K \subset E^\omega$ e.g. if E is a Banach space. Conversely, a *globalization* of a given Harish–Chandra module V is an admissible representation of G with $V = E_K$. Our main result is an independent approach to aspects of Schmid's and Kashiwara's theory of globalizations, which asserts in particular that every Harish-Chandra module admits a unique *minimal globalization*, equivalent to E^ω for all Banach globalizations E.

Theorem ([3])**.** *Let G be a real reductive group. Then every Harish-Chandra module V for G admits a unique $\mathcal{A}(G)$–tempered analytic globalization V^{mg}. Moreover, V^{mg} has the property $V^{mg} = \Pi(\mathcal{A}(G))V$.*

Corollary. *a) $E^\omega \simeq V^{mg}$ for every $\mathcal{A}(G)$-tempered globalization E of V.*
b) For an irreducible admissible Banach representation, E^ω is an algebraically simple $\mathcal{A}(G)$-module.

Schmid constructs his minimal globalization from a realization of V as smooth functions on G using matrix coefficients. His and Kashiwara's results employ the convolution algebra of test functions instead of $\mathcal{A}(G)$ and apply to any complete globalization without temperedness assumptions. In our language they obtain a factorization $E^\omega = \Pi(C_c^\infty(G))E_K$ for every complete globalization E.

Our more explicit approach defines V^{mg} as the quotient of $\mathcal{A}(G)^k$ with its diagonal G-action by the kernel of the G-equivariant continuous map

$$\mathcal{A}(G)^k \to E, \quad (f_1, \dots, f_k) \mapsto \sum_{j=1}^{k} \Pi(f_j)v_j \ ,$$

where V is considered as a subspace generated by $v_1, \dots, v_k$ in an arbitrary $\mathcal{A}(G)$–tempered globalization. V^{mg} is independent of the choice of the globalization and (up to equivalence) of the set of generators and satisfies $V^{\mathrm{mg}} = \Pi(\mathcal{A}(G))V$. It is minimal in the sense that it injects into the space of analytic vectors of any other $\mathcal{A}(G)$–tempered globalization, and its functorial properties are readily deduced from the construction. Topologically V^{mg} turns out to be a reflexive DNF space and is, in particular, complete.

General arguments and the functorial properties reduce the proof of the theorem to the case where the Harish-Chandra module V is the space of K–finite vectors of a spherical principal series representation. The main step is to show that in this situation V^{mg} coincides with the analytic functions on $M\backslash K$ as analytic representations, where $M = Z_K(A)$ for some Iwasawa decomposition $G = KAN$.

The key ingredient in the proof are some recent lower bounds for matrix coefficients obtained in [1]. The techniques developed by Bernstein and Krötz in their article in the context of smooth globalizations can be adapted to yield an explicit factorization of $v \in C^\omega(M\backslash K)$ as $v = \Pi(F)\xi$ for a suitable K-finite $\xi \in V$. The main new difficulty is to show $F \in \mathcal{A}(G)$. To do so, the coincidence of analytic and Δ–analytic vectors for Fréchet representations of moderate growth [2] turns out to be convenient. Both the theorem and the assertion in part b) of the Corollary follow from the thus obtained factorization $C^\omega(M\backslash K) = \Pi(\mathcal{A}(G))V = V^{\mathrm{mg}}$.

References

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