Spectral projections of boundary problems and the regularity of the eta function

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1 Introduction

Consider a differential boundary problem (A, B) of order $m > 0$ on the L^2–sections of a vector bundle E over a manifold with boundary X, and let A_B the realization of A on $L^2(X, E)$ with domain $\mathcal{D}(A_B) = \{u \in H^m(X, E) : Bu = 0\}$. We assume that $(A - \lambda, B)$ is parameter–elliptic for λ in two sectors around $\mathbb{R}_+ e^{i\vartheta}$ resp. $\mathbb{R}_+ e^{i\varphi}$ and that the spectrum is disjoint from these rays. Then $\|(\lambda - A_B)^{-1}\| \lesssim |\lambda|^{-1}$ along these rays, and we may consider the zeta functions $\zeta_\vartheta(z, A, B) = \mathrm{Tr}\ (A_B)_\vartheta^{-z}$, $\zeta_\varphi(z, A, B) = \mathrm{Tr}\ (A_B)_\varphi^{-z}$ associated with the spectral cuts ϑ, φ. They extend to meromorphic functions on $\mathbb{C}$ and are regular in $z = 0$. If A_B is selfadjoint, the eta function $\eta(z, A, B) = \mathrm{Tr}\ A_B |A_B|^{-z-1}$ similarly admits a meromorphic continuation. Both $\mathrm{Res}_{z=0}\eta(z, A, B)$ and $\zeta_\vartheta(0, A, B) - \zeta_\varphi(0, A, B)$ for $\vartheta \neq \varphi$ are given as an explicit integral involving the symbols of A and B, but even on closed manifolds their vanishing is a subtle topological theorem of Atiyah/Patodi/Singer [1], Gilkey [6] and Wodzicki [11]. In particular, the eta invariant $\eta(0)$ of an operator on a closed manifold, which appears as a spectral boundary term in topological index formulas, is finite. Here we establish the corresponding result for manifolds with boundary:

Theorem 1.1. *Under the above assumptions,* $\zeta_\vartheta(0, A, B) = \zeta_\varphi(0, A, B)$.

Theorem 1.2. *If A_B is selfadjoint, the eta function $\eta(z, A, B)$ is regular in $z = 0$.*

For the proof, we construct a "minimal" Fréchet algebra $\mathfrak{A}$ of general 0–order boundary problems not necessarily satisfying the transmission condition. It is inspired by and a subalgebra of Rempel–Schulze's LF algebra [8], but admits a symbol calculus very similar to Boutet de Monvel's algebra $\mathfrak{B}$ [2] including a normal trace $\mathrm{tr_n}$ and a noncommutative residue res. The key observation is that $\mathfrak{A}$ contains the sectorial projections $\Pi_{\vartheta\varphi}(A, B)$ associated to A_B. In particular, $\Pi_{\vartheta\varphi}(A, B)$ belongs to Rempel–Schulze's algebra, verifying a conjecture of Ivrii and, independently, by Gaarde and Grubb.
As $\mathrm{Res}_{z=0}\, \eta(z, A, B) = \frac{i}{\pi}\left(\zeta_{\frac{\pi}{2}}(0, A, B) - \zeta_{-\frac{\pi}{2}}(0, A, B)\right)$ and

$$\zeta_\vartheta(0, A, B) - \zeta_\varphi(0, A, B) = \frac{2\pi i}{m}\,\mathrm{res}(\Pi_{\vartheta\varphi}(A, B)),$$

our results follow from an analogue of Wodzicki's [11] vanishing theorem:

Theorem 1.3. res *vanishes on projections in* $\mathfrak{A}$.

2 An algebra of boundary problems

Let X be a manifold with boundary and $n = \dim X > 1$. For convenience, we will assume that all changes of coordinates preserve the normal coordinate on the collar $\partial X \times [0,1)$, and we will write these local coordinates as (y,t). We also employ the usual notation for operator–valued pseudodifferential operators.

Definition 2.1. Denote by ℓ^1_∞ the space of all bounded operators T on $L^2(\mathbb{R}_+)$ such that for all $k,l \in \mathbb{N}_0$, the operators $[t]^k(t\partial_t)^l T$ and $T[t]^k(t\partial_t)^k$ are densely defined and extend to trace class operators on $L^2(\mathbb{R}_+)$. Here, $t \mapsto [t]$ denotes a strictly positive smooth function on $\mathbb{R}$ such that $[t] = |t|$ for $|t| \geq 1$.

Definition 2.2. (a) $\mathfrak{R}$ is the space of all operators R on $L^2(X)$ that can be written with a C^∞ integral kernel on $X^\circ \times X^\circ$ such that for each $k \in \mathbb{N}_0$ and each cut-off function in a neighborhood of the boundary, σ, the operators $\sigma(t\partial_t)^k R$, $R(t\partial_t)^k\sigma$, are trace class on $L^2(X)$.

(b) $\mathfrak{G}$ is the span of all operators G of the form $G = \sigma_1 H \sigma_2 + R$ where σ_1 and σ_2 are cut-off functions in a neighborhood of the boundary, H is a pseudodifferential operator in $\Psi^0_{cl}(\partial X; \ell^1_\infty)$, and $R \in \mathfrak{R}$.

Lemma 2.3. *For every $G \in \mathfrak{G}$, the L^2–trace $\operatorname{tr}_n G$ in the normal direction defines an element of $\Psi^0_{cl}(\partial X)$.*

Definition 2.4. (a) The Mellin transform $\mathcal{M}u$ of a function $u \in C^\infty_c(\mathbb{R}_+)$ is given by

$$(Mu)(z) = \int_0^\infty t^{z-1} u(t)\, dt.$$

$\mathcal{M}$ extends to an isomorphism $\mathcal{M} : L^2(\mathbb{R}_+) \to L^2(\{z \in \mathbb{C} : \Re z = \frac{1}{2}\})$.

(b) $\mathcal{T}$ is the space of all functions in $L^2(\mathbb{R}_+)$ whose Mellin transform belongs to the Schwartz space $\mathcal{S}(\{\Re z = \frac{1}{2}\})$.

Definition 2.5. (a) Given a cut-off function ω, we define the Mellin operator $M_{\omega,\mu}$ with Mellin symbol $\mu \in C^\infty_b(\mathbb{R}^q, \mathcal{T})$ and cut-off function ω on $\mathcal{S}(\mathbb{R}^q \times \mathbb{R}_+)$ by

$$M_{\omega,\mu} u(y,t) = \int \int_0^\infty e^{iy\eta} \omega(t[\eta]) \mu(y, t/s) (\mathcal{F}_{y\to\eta} u)(\eta, s) \frac{ds}{s}\, d\eta.$$

Here $\mathcal{F}_{y\to\eta}$ denotes the partial Fourier transform with respect to the y-variables.

(b) We denote by $\mathfrak{M}$ the set of all operators on $\partial X \times \mathbb{R}_+$ of the form $M = \sigma_1 \widetilde{M} \sigma_2$, where σ_1 and σ_2 are cut-off functions and, in local coordinates, $\widetilde{M}$ is a Mellin operator with a Mellin symbol μ and a cut-off function σ.

The vector spaces $\mathfrak{M}$ and $\mathfrak{G}$ are endowed with the natural Fréchet topologies.

Definition 2.6. We define

$$\mathfrak{A} = \{A_+ + M + G \ : \ A \in \Psi^0_{cl}(2X), \ M \in \mathfrak{M}, \ G \in \mathfrak{G}\}$$

and endow it with the topology of a non-direct sum of Fréchet spaces.

Theorem 2.7. $\mathfrak{A}$ *is a Fréchet algebra.*

Remark 2.8. The construction extends to give a Fréchet algebra of operators, also denoted by $\mathfrak{A}$, acting on the L^2–sections of a vector bundle E over X.

The crucial differences between $\mathfrak{A}$ and related algebras are the following: We replace the usual LF space of functions with holomorphic Mellin transform in some strip around $\{\Re z = \frac{1}{2}\}$ by the *Fréchet* space $\mathcal{T}$, and use ℓ^1_∞ instead of Hilbert–Schmidt operators to have a normal trace. This adds a number of nontrivial computations to the proof that $\mathfrak{A}$ is an algebra. Furthermore, $\mathfrak{M}$ only includes very special conormally smoothing Mellin operators.

3 Properties of $\mathfrak{A}$ and Wodzicki's theorem

Theorem 3.1. (Properties of $\mathfrak{A}$)

(a) *Operators in* $\mathfrak{A}$ *have natural mapping properties in a scale of logarithmically weighted conormal Sobolev spaces modelled on*

$$\mathcal{H}^{s,k}_{\log}(\mathbb{R}_+) = \{u \in L^2(\mathbb{R}_+) : \ t \mapsto [\log(t)]^k (t\partial_t)^j u(t) \in L^2(\mathbb{R}_+) \ \forall j \leq s\}.$$

(b) *As a subalgebra,* $\mathfrak{A}$ *inherits interior principal and boundary symbol maps as well as a notion of ellipticity from Rempel–Schulze's algebra.*

(c) *Elliptic operators admit a parametrix. Conversely, the Fredholm property implies ellipticity.*

(d) $\mathfrak{A}$ *is spectrally invariant, the set* $\mathfrak{A}^{-1}$ *of invertible elements is open, and inversion is continuous on* $\mathfrak{A}^{-1}$*. Therefore* $K(\mathfrak{A}) = K(\overline{\mathfrak{A}})$*.*

(e) *Similar to an argument in Rempel–Schulze's paper, projections in* $\mathfrak{A}$ *may be deformed to satisfy the transmission condition, whence* $K(\mathfrak{A}) = K(\mathfrak{B})$*.*

Definition 3.2. For $P = A_+ + M + G \in \mathfrak{A}$, define the noncommutative residue

$$\operatorname{res} P = \int_{S^*X^\circ} \operatorname{Tr} \sigma_{2X}(A)_{-n}(x,\xi) \, đ\sigma(\xi) dx + \int_{S^*\partial X} \operatorname{Tr} \ \sigma_{\partial X}(\operatorname{tr_n} G)_{1-n}(x',\xi') \, đ\sigma(\xi') dx'.$$

Here, Tr denotes the pointwise trace in the fibre $\operatorname{End}(E_x, E_x)$.

Remark 3.3. The residue is well-defined because $\mathfrak{M} \cap \mathfrak{G} \subset \Psi^0(\ell^1_\infty)$ consists of operators whose normal trace is 0–homogeneous up to a smoothing remainder and is therefore contained in the kernel of res. This residue extends the residue from Boutet de Monvel's algebra to $\mathfrak{A}$.

Theorem 3.4. *For all* $P, Q \in \mathfrak{A}$*,* $\operatorname{res}([P,Q]) = 0$.

The proof is a lengthy computation replacing the complex analytic/Wiener–Hopf arguments in [3] by distribution theory.

Corollary 3.5. *The residue* res *induces a map* $\overline{\mathrm{res}} : K(\overline{\mathfrak{A}}) \to \mathbb{C}$.

As $K(\overline{\mathfrak{A}}) = K(\mathfrak{A}) = K(\mathfrak{B})$ and $\overline{\mathrm{res}}$ is compatible with the residue on $\mathfrak{B}$, it is sufficient to show $\overline{\mathrm{res}} = 0$ for projections in Boutet de Monvel's algebra. By a theorem of Melo, Nest and Schrohe [7], such a K–class decomposes into a compactly supported class in the interior and the class of an endomorphism in a neighborhood of the boundary. As the latter is homogeneous of order 0, its residue vanishes. The compactly supported class, however, extends to $2X$ where we may apply Wodzicki's theorem. In particular, (a stronger variant for $\overline{\mathfrak{A}}$ of) Theorem 1.3 follows.

4 Spectral projections

For a differential boundary problem (A, B) of order $m > 0$, such that $(A-\lambda, B)$ is parameter–elliptic in two sectors around $\mathbb{R}_+ e^{i\vartheta}$ resp. $\mathbb{R}_+ e^{i\varphi}$ and no eigenvalues lie on these rays, we define the sectorial projection

$$\Pi_{\vartheta\varphi}(A, B) = \frac{i}{2\pi} \int_{\Gamma_{\vartheta\varphi}} \lambda^{-1} A_B (\lambda - A_B)^{-1} \, d\lambda,$$

where $\Gamma_{\vartheta\varphi} = \{re^{i\varphi} : r \in (r_0, \infty)\} \cup \{r_0 e^{i\omega} : \varphi \geq \omega \geq \vartheta\} \cup \{re^{i\vartheta} : r \in (r_0, \infty)\}$. Here, r_0 should be smaller than the smallest absolute value of the nonzero eigenvalues in the sector between φ and ϑ.
We note the following consequence of Seeley's work [9], which is basically contained in Gaarde and Grubb's analysis [5]:

Proposition 4.1. $\Pi_{\vartheta\varphi}(A, B) = \Pi_{\vartheta\varphi}(A)_+ + G^0_{\vartheta\varphi} + \widetilde{G}_{\vartheta\varphi}$*, where* $\Pi_{\vartheta\varphi}(A) \in \Psi^0_{cl}(2X)$ *and* $\widetilde{G}_{\vartheta\varphi} \in \mathfrak{G}$ *admits an expansion into terms whose kernel is quasihomogeneous of order* ≤ -1*.* $G^0_{\vartheta\varphi}$ *has a quasihomogeneous kernel of order* 0 *and is given by* $G^0_{\vartheta\varphi} = \frac{i}{2\pi} \int_{\Gamma_{\vartheta\varphi}} G^0_\lambda \, d\lambda$ *as an integral over the leading quasihomogeneous term* G^0_λ *of the singular Green part of* $(\lambda - A_B)^{-1}$*.*

Our analysis of the leading term $G^0_{\vartheta\varphi}$ implies:

Theorem 4.2. $G^0_{\vartheta\varphi} \in \mathfrak{M} + \mathfrak{G}$ *and* $\Pi_{\vartheta\varphi}(A, B) \in \mathfrak{A}$*.*

To convey the idea of the proof, we provide the key computation which extracts the Mellin part from $G^0_{\vartheta\varphi}$ for scalar problems up to an error in $\Psi^0_{cl}(\partial X, \ell^2(L^2(\mathbb{R}_+)))$. This only shows that $\Pi_{\vartheta\varphi}(A, B)$ belongs to Rempel–Schulze's algebra, but by investing a bit more and quite similar work to verify its mapping properties, the Hilbert–Schmidt term $\ell^2(L^2(\mathbb{R}_+))$ may be improved to ℓ^1_∞.
For simplicity, we only consider the integral $\int_{\Gamma_{\vartheta\varphi}} G^0_\lambda \, d\lambda$ along one ray $\{re^{i\varphi} : r \in (r_0, \infty)\} \subset \Gamma_{\vartheta\varphi}$ and choose $\varphi = 0$. The explicit representation for G^0_λ, which we are going to use, is due to Seeley [9]. First note that for fixed η and large λ, the scalar equation $\sigma(A)(y, 0, \eta, \tau) - \lambda = 0$ for τ has simple roots $i\kappa^+_k(\eta, \lambda)$ resp. $-i\kappa^-_l(\eta, \lambda)$ in the upper resp. lower half–planes.

In the following, we are going to keep η fixed. Performing some straight–forward contour integrals in Seeley's formulas, for large λ the integral kernel of $G^0_\lambda(\eta,t,s)$ in the normal variables $s,t\in\mathbb{R}_+$ turns out to be a sum of terms of the form

$$\frac{e^{-\kappa^+_k(\eta,\lambda)t-\kappa^-_l(\eta,\lambda)s}}{\sigma_{kl}(\eta,\lambda)}\;\widetilde{\sigma}_{kl}(\eta,\lambda),$$

where $\sigma_{kl},\widetilde{\sigma}_{kl}$ are homogeneous functions in $(\eta,\lambda^{1/m})$ of degrees $m-1$ and 0 (obviously, we might as well set $\widetilde{\sigma}_{kl}=1$ in our situation). The crucial idea is to approximate this kernel by its asymptotic behavior for large λ. Homogeneity gives

$$\begin{aligned}\kappa^\pm_k(\eta,\lambda) &= \lambda^{1/m}\kappa^\pm_k(0,+1)+\mathcal{O}(\lambda^{1/m-\varepsilon}),\\ \sigma_{kl}(\eta,\lambda) &= \lambda^{(m-1)/m}\sigma_{kl}(0,+1)+\mathcal{O}(\lambda^{(m-1)/m-\varepsilon}),\\ \widetilde{\sigma}_{kl}(\eta,\lambda) &= \widetilde{\sigma}_{kl}(0,+1)+\mathcal{O}(\lambda^{-\varepsilon})\end{aligned}$$

for some $\varepsilon>0$. Setting $K^\pm_k=\kappa^\pm_k(0,+1)$, $\Sigma_{kl}=\sigma_{kl}(0,1)$, and $\widetilde{\Sigma}_{kl}=\widetilde{\sigma}_{kl}(0,1)$, we define

$$M^R_{kl}(t,s) \;=\; \int_R^\infty \frac{e^{-\lambda^{1/m}(K^+_k t+K^-_l s)}}{\Sigma_{kl}\;\lambda^{(m-1)/m}}\;\widetilde{\Sigma}_{kl}\;d\lambda$$

for sufficiently large $R>0$. That M^R_{kl} actually exists, is a consequence of the ellipticity assumption.
We estimate the Hilbert–Schmidt norm of the error

$$\left\|\int_{r_0}^\infty G^0_\lambda(\eta,t,s)\;d\lambda-\sum_{k,l}M^R_{kl}(t,s)\right\|_{L^2(\mathbb{R}_+\times\mathbb{R}_+)} \;\le\; \left\|\int_{r_0}^R G^0_\lambda(\eta,t,s)\;d\lambda\right\|_{L^2(\mathbb{R}_+\times\mathbb{R}_+)} +$$
$$\sum_{k,l}\left\|\int_R^\infty\left\{\frac{e^{-\kappa^+_k(\eta,\lambda)t-\kappa^-_l(\eta,\lambda)s}}{\sigma(\eta,\lambda)}\;\widetilde{\sigma}(\eta,\lambda)-\frac{e^{-\lambda^{1/m}(K^+_k t+K^-_l s)}}{\Sigma_{kl}\;\lambda^{(m-1)/m}}\;\widetilde{\Sigma}_{kl}\right\}d\lambda\right\|_{L^2(\mathbb{R}_+\times\mathbb{R}_+)}.$$

Using the triangle inequality, each term in the sum is dominated by

$$\int_R^\infty\left(\int_0^\infty\int_0^\infty\left(\frac{e^{-\kappa^+_k(\eta,\lambda)t-\kappa^-_l(\eta,\lambda)s}}{\sigma(\eta,\lambda)}\;\widetilde{\sigma}(\eta,\lambda)-\frac{e^{-\lambda^{1/m}(K^+_k t+K^-_l s)}}{\Sigma_{kl}\;\lambda^{(m-1)/m}}\;\widetilde{\Sigma}_{kl}\right)^2dt\;ds\right)^{1/2}d\lambda,$$

and a second straight forward computation shows the finiteness of this integral. Hence, $\int_{r_0}^\infty G^0_\lambda(\eta,t,s)\;d\lambda-\sum_{k,l}M^R_{kl}(t,s)$ is the kernel of a Hilbert–Schmidt operator on $L^2(\mathbb{R}_+)$.
On the other hand, M^R_{kl} may be evaluated explicitly:

$$M^R_{kl}(t,s) \;=\; \frac{m\widetilde{\Sigma}_{kl}}{\Sigma_{kl}}\;\frac{e^{-R^{1/m}(K^+_k t+K^-_l s)}}{K^+_k t+K^-_l s}.$$

Up to another Hilbert–Schmidt operator on $L^2(\mathbb{R}_+)$, M^R_{kl} is thus a Mellin operator with symbol $\mu_{kl}(\tau)=\frac{m\widetilde{\Sigma}_{kl}}{\Sigma_{kl}}\;\frac{1}{K^+_k\tau+K^-_l}$, which is independent of R.

As mentioned above, investigating the mapping properties of $\int_{r_0}^{\infty} G^0_\lambda(\eta, t, s)\, d\lambda - \sum_{k,l} M^R_{kl}$ gives the stronger assertion of Theorem 4.2. The method also extends to the nonscalar case, where the Mellin symbols turn out to be more complicated rational functions.
The relation between $\zeta_\vartheta(0, A, B) - \zeta_\varphi(0, A, B)$ and $\mathrm{res}(\Pi_{\vartheta\varphi}(A, B))$ mentioned in the introduction is now deduced from the explicit formulas in Seeley's work [10]. Theorems 1.1 and 1.2 then follow from Theorems 4.2 and 1.3.

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