Appendix 1
Wave Fronts and Propagation of Singularities

In this appendix we present the definition and the simplest properties of the wave front of a distribution as introduced by Hörmander [6]. The concept of wave front is important in that it allows a microlocal (localized at a point of the cotangent bundle) formulation of the theorems on regularity of solutions of differential equations and also clarifies questions connected with the propagation of singularities. The wave fronts also play an important role on spectral theory, and are naturally connected with pseudodifferential operators. While leaving out many important questions of the theory of wave fronts, *I* nevertheless thought it useful to add this short appendix.

A.1.1 Wave front of a distribution

Definition A.1.1. Let X be an open set in $\mathbb{R}^n$, let $(x_0, \xi_0) \in X \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathscr{D}'(X)$. We shall write $(x_0, \xi_0) \notin WF(u)$ if there exists $v \in \mathscr{E}'(X)$ such that $u = v$ in a neighborhood of x_0 and

$$|\tilde{v}(\xi)| \leqq C_N \langle\xi\rangle^{-N} \quad \text{if} \quad \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \varepsilon, \tag{A.1.1}$$

for sufficiently small $\varepsilon > 0$ and arbitrary $N > 0$, i.e. $\tilde{v}(\xi)$ is rapidly decreasing in a conic neighbourhood of ξ_0.

Thus $WF(u)$ is a closed conic set in $X \times (\mathbb{R}^n \setminus \{0\})$ which we call *the wave front of the distribution u.*

Lemma A.1.1. *If* $\varphi(x) \in C_0^\infty(X)$ *and* $(x_0, \xi_0) \notin WF(u)$ *then* $(x_0, \xi_0) \notin WF(\varphi u)$.

Proof. We have to show that if $\tilde{v}$ is rapidly decreasing in an open cone Γ, then so is $\widetilde{\varphi v}$. Now

$$\begin{aligned}\widetilde{\varphi v}(\xi) &= \int \tilde{v}(\xi - \eta)\, \tilde{\varphi}(\eta)\, d\eta \\ &= \int_{|\eta| \leqq R} \tilde{v}(\xi - \eta)\, \tilde{\varphi}(\eta)\, d\eta + \int_{|\eta| \geqq R} \tilde{v}(\xi - \eta)\, \tilde{\varphi}(\eta)\, d\eta\end{aligned}$$

and

$$\begin{aligned}|\widetilde{\varphi v}(\xi)| &\leqq C \sup_{|\eta|\leqq R} |\tilde v(\xi-\eta)| + C_L \int_{|\eta|\geqq R} (1+|\xi-\eta|)^p (1+|\eta|)^{-L} d\eta \\ &\leqq C \sup_{|\eta|\leqq R} |\tilde v(\xi-\eta)| + C_L (1+|\xi|)^p \int_{|\eta|\geqq R} (1+|\eta|)^{p-L} d\eta \\ &\leqq C \sup_{|\eta|\leqq R} |\tilde v(\xi-\eta)| + C_L (1+|\xi|)^p R^{n+p-L}.\end{aligned}$$

Putting $R=|\xi|^{1/2}$, we see that if ξ belongs to a cone slightly smaller than Γ, then $\xi-\eta\in\Gamma$ for large $|\xi|$ and $|\eta|\leqq R$. In addition, $|\xi-\eta|\sim|\xi|$ and $R^{n+p-L} \sim |\xi|^{(n+p-L)/2}$. Picking a large L we see that $\widetilde{\varphi v}(\xi)$ is rapidly decreasing in ξ as $|\xi|\to\infty$, $\xi\in\Gamma$. □

Corollary A.1.1. *In Definition A.1.1 we may put* $v=\varphi u$, *for* $\varphi\in C_0^\infty(X)$.

Proof. We may choose first v as in the definition and then φ such $\varphi=1$ in a neighborhood of x_0 and $\varphi u=\varphi v$. It remains only to apply Lemma A.1.1. □

Lemma A.1.2. Let π: $X\times(\mathbb{R}^n\setminus 0)\to X$ be the natural projection and $u\in\mathscr{D}'(X)$. Then

$$\pi WF(u) = \operatorname{sing\,supp} u.$$

Proof. a) If $x_0\notin\operatorname{sing\,supp} u$ pick $\varphi\in C_0^\infty(X)$ such that $\varphi=1$ in a neighbourhood of x_0, $\varphi=0$ in a neighbourhood of sing supp u. Then we see that $\varphi u\in C_0^\infty(X)$, from which $\varphi u\in S(\mathbb{R}^n)$ i.e. $x_0\notin\pi WF(u)$.

b) Let $x_0\notin\pi WF(u)$. Then for any $\xi_0\in\mathbb{R}^n\setminus\{0\}$ there exists a function $\varphi_{\xi_0}(x)\in C_0^\infty(X)$ and a conical neighbourhood Γ_{ξ_0} of ξ_0 such that $\varphi_{\xi_0}(x)=1$ near x_0 and $(\widetilde{\varphi_{\xi_0}u})(\xi)$ decreases rapidly in Γ_{ξ_0}. Let $\Gamma_{\xi_1},\ldots,\Gamma_{\xi_N}$ be a covering of $\mathbb{R}^n\setminus\{0\}$. Putting $\varphi=\prod_{j=1}^N \varphi_{\xi_j}$ we see that $\widetilde{\varphi u}(\xi)$ decreases rapidly everywhere so that $\varphi u\in C_0^\infty(X)$ i.e. $u\in C^\infty$ in a neighbourhood of x_0 so $x_0\notin\operatorname{sing\,supp} u$.

Proposition A.1.1. *Let* $u\in\mathscr{D}'(X)$ *and* $(x_0,\xi_0)\notin WF(u)$. *Then there exists a classical properly supported* ΨDO $A\in CL^0(X)$ *such that* $\sigma_A\equiv 1 \pmod{S^{-\infty}}$ *in a conic neighbourhood of* (x_0,ξ_0) *and* $Au\in C_0^\infty(X)$.

Proof. Let $\varphi\in C_0^\infty(X)$, $\varphi=1$ in a neighbourhood of x_0 and suppose $\widetilde{\varphi u}(\xi)$ decreases rapidly in a conic neighbourhood of ξ_0. Let $\chi(\xi)$ be supported in this neighbourhood with $\chi(t\xi)=\chi(\xi)$ for $t\geqq 1$, $|\xi|\geqq 1$ and $\chi(\xi)\in C^\infty(X)$ with $\chi(\xi)=1$ in some smaller conic neighbourhood of ξ_0. Then $\chi(\xi)\,\widetilde{\varphi u}(\xi)$ decreases rapidly so that $\chi(D)(\varphi(x)u(x))\in C^\infty$. But then $\psi(x)\chi(D)(\varphi(x)u(x))\in C_0^\infty(X)$ if $\psi\in C_0^\infty(X)$. We may pick ψ so that $\psi(x)=1$ in a neighbourhood of x_0 and then the ΨDO $A=\psi(x)\chi(D)\varphi(x)$ satisfies all the required conditions. □

Proposition A.1.2. *Suppose we are given* $u\in\mathscr{D}'(X)$, $(x_0,\xi_0)\in X\times(\mathbb{R}^n\setminus\{0\})$ *and an operator* $A\in CL^m(X)$ *with principal symbol* $a_m(x,\xi)$. *So that* Au *makes* *sense let either* $u \in \mathscr{E}'(X)$ *or* A *be properly supported. Finally assume that* $a_m(x_0, \xi_0) \neq 0$ *and* $Au \in C^\infty(X)$. *Then* $(x_0, \xi_0) \notin WF(u)$.

Proof. a) By the standard construction of a parametrix in a conic neighbourhood of (x_0, ξ_0) (cf. §5), we obtain a properly supported ΨDO $B \in CL^{-m}(X)$ such that $\sigma_{BA} \equiv 1$ $(mod S^{-\infty})$ in this neighbourhood. Obviously $BAu \in C^\infty(X)$ so that replacing A by BA we could obtain $A = I (\operatorname{mod} S^{-\infty})$ in the same conic neighbourhood of the point (x_0, ξ_0).

b) Now let $\chi(\xi) = 1$ in a neighbourhood of ξ_0, $\chi(\xi) \in C^\infty(\mathbb{R}^n)$, and $\chi(\xi)$ homogeneous of order 0 in ξ for $|\xi| \geqq 1$. Let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 1$ in a neighbourhood of x_0 and the supports of φ and χ be chosen so that

$$\varphi(x)\,\chi(\xi)\,\sigma_A(x, \xi) = \varphi(x)\,\chi(\xi) \qquad (\operatorname{mod} S^{-\infty}).$$

From this we obtain

$$\chi(D)\,\varphi(x) A - \chi(D)\,\varphi(x) \in L^{-\infty}, \tag{A.1.2}$$

and by $\chi(D)\,\varphi(x) Au \in C^\infty(X)$, it follows from (A.1.2) that

$$\chi(D)\,\varphi(x) u \in C^\infty(\mathbb{R}^n). \tag{A.1.3}$$

c) Now we show that

$$\chi(D)\,\varphi(x)\, u \in S(\mathbb{R}^n), \tag{A.1.4}$$

from which it follows that $\chi(\xi)\,\widetilde{\varphi u}(\xi) \in S(\mathbb{R}^n)$ and in particular, that $\widetilde{\varphi u}(\xi)$ decreases rapidly in a conic neighbourhood of ξ_0 as required. (A.1.4) follows from the following lemma and (A.1.3)

Lemma A.1.3. *Let* $v \in \mathscr{E}'(\mathbb{R}^n)$, $\chi(\xi) \in S^m_{\varrho,0}$. *Then for* $\varrho(x, \operatorname{supp} v) \geqq 1$ *we have*

$$|D^\alpha \chi(D)\, v(x)| \leqq C_{\alpha,N} |x|^{-N} \tag{A.1.5}$$

Proof. Since $\xi^\alpha \chi(\xi) \in S^{m+|\alpha|}_{\varrho,0}$ it all reduces to the case $\alpha = 0$. Further since $v = \sum_{|\alpha| \leqq p} D^\alpha v_\alpha$ with continuous v_α, we may reduce to the situation where v is continuous.

We have

$$\chi(D)\, v(x) = \int e^{i(x-y)\cdot\xi}\, \chi(\xi)\, v(y)\, dy\, đ\xi. \tag{A.1.6}$$

Integrating by parts and using the formula

$$|x-y|^{-2N} (-\Delta_\xi)^N\, e^{i(x-y)\cdot\xi} = e^{i(x-y)\cdot\xi},$$

from (A.1.6) we obtain

$$\chi(D)\, v(x) = \int e^{i(x-y)\cdot\xi} ((-\Delta_\xi)^N \chi(\xi))\, |x-y|^{-2N}\, v(y)\, dy\, đ\xi, \tag{A.1.7}$$

which makes sense for $\varrho(x, \operatorname{supp} v) \geqq 1$. Picking N so large that $(-\Delta_\xi)^N \chi(\xi) \in S_{\varrho,0}^{-n-1}$, we see that the integral (A.1.7) converges absolutely and is estimated by $C\langle x\rangle^{-2N}$ for $\varrho(x, \operatorname{supp} v) \geqq 1$. □

Remark A.1.1. The condition $a_m(x_0, \xi_0) \neq 0$ is sometimes called *ellipticity of A at* (x_0, ξ_0). It is easy to formulate and prove the hypoelliptic analogue of Proposition A.1.2. We leave this for the reader as an excercise.

Corollary A.1.2. *For $A \in CL^m(X)$ denote*

$$\operatorname{char}(A) = \{(x, \xi) \in X \times (\mathbb{R}^n \setminus 0) \colon a_m(x, \xi) = 0\}.$$

Then if $Au = f \in C^\infty(X)$ we have $WF(u) \subset \operatorname{char}(A)$. In particular, if $\operatorname{char}(A) = \emptyset$ *we have $u \in C^\infty(X)$.*

Corollary A.1.3. *If $u \in \mathscr{E}'(X)$, then*

$$WF(u) = \bigcap_{\substack{A \in CL^0(X) \\ Au \in C^\infty(X)}} \operatorname{char}(A). \tag{A.1.8}$$

This holds for $u \in \mathscr{D}'(X)$ if we take the intersection only over properly supported A.

The importance of Corollary A.1.3 is that (A.1.8) shows how to define $WF(u)$ invariantly as a closed conic subset of T^*X when X is a manifold. We now generalize Proposition A.1.2 even more, by weakening the requirement $Au \in C^\infty(X)$.

Proposition A.1.3. *Again let $A \in CL^m(X)$, $u \in \mathscr{D}'(X)$ and either A be properly supported or $u \in \mathscr{E}'(X)$. Then, assuming $a_m(x_0, \xi_0) \neq 0$ and $(x_0, \xi_0) \notin WF(Au)$, we have $(x_0, \xi_0) \notin WF(u)$. In other words,*

$$WF(u) \subset \operatorname{char}(A) \cup WF(Au). \tag{A.1.9}$$

Proof. By proposition A.1.1 there exists a properly supported $P \in CL^0(X)$, with $\sigma_P \equiv 1 \pmod{S^{-\infty}}$ in a conic neighbourhood of (x_0, ξ_0) and $(PA)(u) \in C^\infty(X)$. But then, from Proposition A.1.2 it obviously follows that $(x_0, \xi_0) \notin WF(u)$. □

Proposition A.1.4 (Pseudolocality of ΨDO). *Let $u \in \mathscr{D}'(X)$, $A \in L^m_{\varrho,\delta}(X)$ $0 \leqq \delta < \varrho \leqq 1$ and assume either A is properly supported or $u \in \mathscr{E}'(X)$. Then if $(x_0, \xi_0) \notin WF(u)$, we have $(x_0, \xi_0) \notin WF(Au)$. In other words,*

$$WF(Au) \subset WF(u). \tag{A.1.10}$$

Proof. The condition $(x_0, \xi_0) \notin WF(u)$ amounts to the existence of a properly supported ΨDO $P \in CL^0(X)$ such that $Pu \in C^\infty(X)$ and $\sigma_P \equiv 1 \pmod{S^{-\infty}}$ in a conic neighbourhood of (x_0, ξ_0). Now let Q be a properly supported ΨDO

$Q \in CL^0(X)$ such that $q_0(x_0, \xi_0) \neq 0$, (q_0 be principal symbol of Q) $\sigma_Q \in S^{-\infty}$ outside of some sufficiently small conic neighbourhood of (x_0, ξ_0) and

$$PQ \equiv Q (\operatorname{mod} L^{-\infty}) \quad \text{if} \quad QP \equiv Q (\operatorname{mod} L^{-\infty}).$$

Let us demonstrate that $QAu \in C^\infty(X)$. We have $QA - QAP \in L^{-\infty}$ so it suffices to show $QAPu \in C^\infty(X)$. This however, is obvious since $Pu \in C^\infty(X)$. Now $(x_0, \xi_0) \notin WF(Au)$ follows from Proposition A.1.2. □

Corollary A.1.4. *If $A \in CL^m(X)$, then*

$$WF(Au) \subset WF(u) \subset WF(Au) \cup \operatorname{char}(A). \tag{A.1.11}$$

Corollary A.1.5. *If the operator $A \in CL^m(X)$ is elliptic, then*

$$WF(Au) = WF(u) \tag{A.1.12}$$

Exercise A.1.1. Compute the wave fronts of the following distributions:

a) $\delta(x)$;

b) $\delta(x') \oplus 1(x'')$, $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$;

c) δ_S, where S is a smooth submanifold in $\mathbb{R}^n$ ($\langle \delta_S, \varphi \rangle$ is defined as the integral of the function φ restricted to the surface S with respect to the induced measure);

d) $(x + i0)^{-1}$ on $\mathbb{R}^1$;

e) the indicator function of an angle in $\mathbb{R}^2$ (the function which is equal to 1 in the angle and 0 outside it).

A.1.2 Applications: Product of two distributions, trace of a distribution on a submanifold

1) Let $u_j \in \mathscr{D}'(X)$, $j = 1, 2$. What does $u_1 \cdot u_2$ mean? It should be the ordinary product $u_1 \cdot u_2$ if, for example, u_1 and u_2 are continuous or if one of them is smooth, and it should be a natural extension (e.g. by continuity in some sense). It will turn out that we can define $u_1 \cdot u_2$ under the condition

$$WF(u_1) + WF(u_2) \subset X \times (\mathbb{R}^n \setminus 0), \tag{A.1.13}$$

i.e. if there are no $(x, \xi) \in WF(u_1)$ such that $(x, -\xi) \in WF(u_2)$.

Since the product is a bilinear operation, then using a partition of unity we may assume that u_1, u_2 are in $\mathscr{E}'(\mathbb{R}^n)$ and have sufficiently small supports so that $\tilde{u}_1(\xi)$ is rapidly decreasing outside a cone Γ_1 and $\tilde{u}_2(\xi)$ is rapidly decreasing outside a cone Γ_2, whereby $\Gamma_1 + \Gamma_2 \subset \mathbb{R}^n \setminus \{0\}$ (i.e. Γ_1 and Γ_2 do not contain opposite points). In the usual situation one has $\widetilde{u_1 u_2}(\xi) = (\tilde{u}_1 * \tilde{u}_2)(\xi)$, where

$$\tilde{u}_1 * \tilde{u}_2(\xi) = \int \tilde{u}_1(\xi - \eta)\, \tilde{u}_2(\eta)\, d\!\!{}^-\eta. \tag{A.1.14}$$

In our case this integral converges absolutely also, since either $|\tilde{u}_1(\xi-\eta)|$ or $|\tilde{u}_2(\eta)|$ rapidly tends to zero as $|\eta|\to+\infty$. Put

$$u_1 u_2 = F^{-1} \int \tilde{u}_1(\xi-\eta)\, \tilde{u}_2(\eta)\, d\eta\,. \tag{A.1.15}$$

Why is this an extension by continuity? We can for instance show that if $\chi\in C_0^\infty(\mathbb{R}^n)$, $\int\chi(x)\,dx=1$, $\chi(x)=\varepsilon^{-n}\chi(\varepsilon^{-1}x)$ and $u^{(\varepsilon)}=u*\chi_\varepsilon(\in C_0^\infty(\mathbb{R}^n))$, then $\lim\limits_{\varepsilon\to 0+} u_1^{(\varepsilon)}u_2^{(\varepsilon)}=u_1u_2$ in the topology of $\mathscr{D}'(\mathbb{R}^n)$ where u_1u_2 is understood in the sense of (A.1.15).

Example A.1.1. Let $u_k\in\mathscr{D}'(\mathbb{R}^2)$,

$$\langle u_k,\varphi\rangle = \int\chi_k(x_1)\,\varphi(x_1,kx_1)\,dx_1\,,\qquad \chi_k(t)\in C_0^\infty(\mathbb{R}^1),$$

so that $\operatorname{supp} u_k\subset\{(x_1,x_2): x_2=kx_1\}$. Consider the product $u_k\cdot u_0$. We have

$$\tilde{u}_k(\xi) = \int\chi_k(x_1)\, e^{i(\xi_1x_1+\xi_2kx_1)}\,dx_1 = \tilde{\chi}_k(\xi_1+k\xi_2)\,.$$

From this we see that $WF(u_k)$ is the set of all normals to the line $x_2=kx_1$, lying over $\pi_k^{-1}(\operatorname{supp}\chi_k)$, where $\pi_k\colon (x_1,kx_1)\to x_1$. Consider now the convolution $\tilde{u}_k*\tilde{u}_0$:

$$\begin{aligned}(\tilde{u}_0*\tilde{u}_k)(\xi) &= \int\tilde{\chi}_0(\xi_1-\eta_1)\,\tilde{\chi}_k(\eta_1+k\eta_2)\,d\eta_1\,d\eta_2\\ &= \int\tilde{\chi}_0(\eta_1)\,d\eta_1\cdot\int\tilde{\chi}_k(k\eta_2)\,d\eta_2 = \frac{1}{k}\,\chi_0(0)\,\chi_k(0);\end{aligned}$$

from which

$$u_0\,u_k = \frac{1}{k}\,\chi_0(0)\cdot\chi_k(0)\cdot\delta(x)\,.$$

The limit as $k\to 0$ does not exist, which is natural, since for $k=0$ condition (A.1.13) fails.

2) Let $u\in\mathscr{D}'(X)$, Y a submanifold of X, NY the family of all normals to Y in T^*X (the normal bundle to Y.) If $WF(u)\cap NY=\emptyset$, then the trace $u|_Y$ is defined naturally in the same sense as for products. Indeed, localizing we may assume that $Y=\{x_1=\ldots=x_k=0\}$. For $u\in C_0^\infty(X)$ we have

$$\widetilde{u|_Y} = \int\tilde{u}(\xi_1,\ldots,\xi_n)\,d\xi_1\ldots d\xi_k. \tag{A.1.16}$$

But by hypothesis vectors of the form $(\xi_1,\ldots,\xi_k,0,\ldots,0)$ are not contained in $WF(u)$, so $\tilde{u}$ decrease rapidly in their direction and the integral (A.1.16) is defined. In particular, if $u\in\mathscr{D}'(\mathbb{R}^{n+1})$ satisfies a differential equation of order m of the form

$$a\left(t,x,\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)u=f\in C^\infty(\mathbb{R}^{n+1}),\qquad t\in\mathbb{R}^1,\ x\in\mathbb{R}^n,$$

where the hyperplane $t=0$ is non-characteristic, i.e. $a_m(0,x,1,0) \neq 0$ then all the restrictions $\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0}$ are defined and lie in $\mathscr{D}'(\mathbb{R}^n)$. This means in particular that the Cauchy data make sense. Thereby $u|_{t=t_0}$ is a smooth function of t_0 with values in $\mathscr{D}'(\mathbb{R}^n)$.

A.1.3 The theorem on propagation of singularities

First, we state the simplest version of the theorem on propagation of singularities.

Theorem A.1.1. *Let $P \in CL^m(X)$ have a real principal symbol $p_m(x,\xi)$, $u \in \mathscr{D}'(X)$ and either P is properly supported or $u \in \mathscr{E}'(X)$ so that Pu makes sense. Then if I is any connected interval of a bicharacteristic of the function $p_m(x,\xi)$ not intersecting $WF(Pu)$ then either $I \subset WF(u)$ or $I \cap WF(u) = \emptyset$.*

In other words, in the complement of $WF(Pu)$, the set $WF(u)$ is invariant under the shifts along the trajectories of the Hamiltonian system

$$\begin{cases} \dot{\xi} = -\dfrac{\partial p_m}{\partial x}, \\ \dot{x} = \dfrac{\partial p_m}{\partial \xi}. \end{cases} \tag{A.1.17}$$

Example A.1.2. Let us prove that the wave equation $\frac{\partial^2 u}{\partial x_0^2} - \Delta u = 0$ in $\mathbb{R}^{n+1}$ cannot have solutions with isolated or compactly supported singularities. We have $m=2$, $p_2(x,\xi) = -\xi_0^2 + |\xi|^2$. The system (A.1.17) has a solution $\xi = \text{const}$, $x_0 = -2\xi_0 t$, $x = 2\xi t$. Let $0 \in \operatorname{sing\,supp} u$. Then there exists a point $(0,0,\xi_0,\xi) \in WF(u)$, so by Proposition A.1.2 it is obvious that $|\xi|^2 = \xi_0^2$. By Theorem A.1.1. $(-2\xi_0 t,\ 2\xi t,\ \xi_0, \xi) \in WF(u)$ for any t and, in particular, $(-2\xi_0 t, 2\xi t) \in \operatorname{sing\,supp} u$ for any t which also yields the required result.

We now give a proof of Theorem A.1.1. due to V.N. Tulovsky. It is based on the following proposition describing wave fronts in terms of the action of distributions on rapidly oscillating exponentials.

Proposition A.1.5. *Let $u \in \mathscr{D}'(X)$. Then the condition $(x_0, \xi_0) \notin WF(u)$ is equivalent to the following condition;*

A. *There exists $\varepsilon > 0$ such that if $\Phi(x,\theta)$ is a smooth real valued function defined for $|x - x_0| < \varepsilon$ and $\theta \in (\mathbb{R}^N \setminus 0)$, is a homogeneous function in θ of degree 1 such that $\Phi_x(x_0, \theta_0) = \xi_0$ for some $\theta_0 \neq 0$, then for an arbitrary symbol $\varphi \in CS^0(\{|x - x_0| < \varepsilon\} \times \mathbb{R}^N)$ vanishing for $|x - x_0| \geqq \varepsilon/2$ there exists a conical neighbourhood Γ of the point θ_0 in $\mathbb{R}^N \setminus 0$ such that*

$$|\langle u(x), \varphi(x,\theta) e^{-i\Phi(x,\theta)} \rangle| \leqq C_N |\theta|^{-N}, \qquad \theta \in \Gamma,\ |\theta| \geqq 1. \tag{A.1.18}$$