some small conic neighborhood Γ of (x_0, ξ_0), it thus follows that for small t, the $c_j(t)$ can be extended by homogeneity as symbols $c_j(t) \in S^{-j}_{\text{comp}}(\Gamma)$, and the symbol $c(t, x, \xi)$ is then constructed as in Lemma 2.2. Since we assume that the bicharacteristic curve is not locally contained in WFu, there is a point $(x(s), \xi(s)) \notin WFu$ for an s in the domain of definition of c, and one has

$$\text{supp } c(s) \subset \text{ supp } c_0(s) \subset e^{sH_q} \text{ supp } c_0$$

where e^{sH_q} supp c_0 denotes the image of supp c_0 under the flow of the Hamiltonian vector field H_q. Therefore, if the (conic) support of c_0 has been chosen sufficiently small around (x_0, ξ_0), the support of $c(s)$ will be small around $(x(s), \xi(s))$ and we will have $w(s) = c(s, x, D)v = \chi \in C^\infty$ (and actually $\chi \in C_0^\infty$ since the support of $c(s)$ is compact in x).

Thanks to Lemma 1.16, the distribution ψu is in some Sobolev space so that $v = \lambda^{m-1}(D)(\psi u) \in H^{-N}$ for some $N \in \mathbb{Z}_+$ and then $w(t) = c(t, x, D)v \in C^0(I; H^{-N})$ since $c \in C^0(I; S^0)$. Moreover, this distribution $w(t)$ satisfies $\partial_t w(t) - ib(x, D)w(t) = g(t)$ where

$$g(t) = r(t, x, D)v - ic(t, x, D)b(x, D)v$$

by construction of $c(t)$, and these two terms are in $C^0(I; \mathcal{S})$ because $r \in C^0(I; S^{-\infty})$ and $v = \lambda^{m-1}(D)(\psi u)$ with $\psi u \in \mathcal{E}'$ (cf. Corollary 3.8), and $b(x, D)v = \varphi f \in C_0^\infty$. It follows that w is a $C^0(I; H^{-N})$ solution of a Cauchy problem as in Lemma 4.9 (indeed, $b \in S^1$ satisfies $b - b^* \in S^0$ since it is polyhomogeneous with a real principal symbol), therefore $w \in \cap_k C^0(I; H^k)$, then $c_0(x, D)v = w(0) \in H^\infty \subset C^\infty$ as claimed. ∎

4.3 The Cauchy problem for the wave equation

This section actually has little to do with pseudodifferential operators. Following an idea of Treves [12], the Cauchy problem for the wave equation is treated through very simple computations, which allow us to prove the main results. We have added this section because the last theorem provides a pleasant illustration of the general results given in the previous section.

Consider in the space $\mathbb{R}^{n+1} = \{(t, x); t \in \mathbb{R}, x \in \mathbb{R}^n\}$ the wave operator, i.e., the operator $u \mapsto \partial_t^2 u - \Delta u$ where $\Delta = \sum_{j=1}^n \partial_j^2$ with $\partial_j = \partial/\partial x_j$ is the Laplacian operator in $\mathbb{R}^n$. The present treatment will be completely dissymmetric in x and t: we will consider solutions u such that for each fixed t, $u(t) \in \mathcal{S}'$ (space of temperate distributions in x only).

The spaces $C^k(\mathbb{R}; H^s)$ and $C^k(\mathbb{R}; \mathcal{S})$ are the standard spaces of C^k functions of t valued in the Hilbert space H^s (Sobolev space in x only) or in the Fréchet space $\mathcal{S}$ (Schwartz space in x only). We will write $u \in C^0(\mathbb{R}; \mathcal{S}')$ if for each fixed $\varphi \in \mathcal{S}$ the expression $(u(t), \varphi)$ is a continuous function of t, and actually, according to general results from functional analysis, these separate continuities imply the continuity of $u : \mathbb{R} \times \mathcal{S} \ni (t, \varphi) \mapsto (u(t), \varphi)$. As a consequence of this remark, if $u \in C^0(\mathbb{R}; \mathcal{S}')$, the formula

$$(U(t), \varphi) = \int_0^t (u(s), \varphi)\, ds \qquad \text{for } \varphi \in \mathcal{S}$$

defines a $U \in C^0(\mathbb{R}; \mathcal{S}')$ also. Then, for $k > 0$ we will write $u \in C^k(\mathbb{R}; \mathcal{S}')$ if for each fixed $\varphi \in \mathcal{S}$, $(u(t), \varphi)$ is a differentiable function of t such that the formula

$$(\partial_t u(t), \varphi) = \frac{d}{dt}(u(t), \varphi)$$

defines a distribution $\partial_t u \in C^{k-1}(\mathbb{R}; \mathcal{S}')$. It is clear that the previous $U(t) = \int_0^t u(s)\, ds$ satisfies $U \in C^{k+1}(\mathbb{R}; \mathcal{S}')$ if $u \in C^k(\mathbb{R}; \mathcal{S}')$.

Most of the proofs of this section will be based on the following, very elementary lemma on trigonometric functions.

LEMMA 4.10
The functions

$$C(t, \xi) = \cos t|\xi| \qquad \textit{and} \qquad S(t, \xi) = \frac{\sin t|\xi|}{|\xi|}$$

are smooth on $\mathbb{R}^{n+1}$ *and for any* $k \in \mathbb{Z}_+$ *and fixed* t, *one has* $\partial_t^k C(t) \in \mathcal{P}$ *and* $\partial_t^k S(t) \in \mathcal{P}$ *as functions of* $\xi \in \mathbb{R}^n$. *Moreover, they satisfy* $\partial_t C(t) = (1 - \lambda^2) S(t)$ *and* $\partial_t S(t) = C(t)$, *they can both be extended to the whole of* $\mathbb{C}^n$ *as entire functions with the estimates*

$$|C(t, \zeta)| \le e^{|t|\,|\operatorname{Im} \zeta|} \qquad \textit{and} \qquad |S(t, \zeta)| \le |t| e^{|t|\,|\operatorname{Im} \zeta|},$$

and finally they also satisfy (for $\xi \in \mathbb{R}^n$*)*

$$|C(t)| \le 1 \qquad \textit{and} \qquad \lambda |S(t)| \le (1 + t^2)^{1/2}.$$

PROOF Indeed, we can write $C(t, \xi) = \sum_{k \ge 0} (-t^2 |\xi|^2)^k / (2k)!$ so that C is obviously smooth and can be extended as an entire function by setting $C(t, \zeta) = \sum_{k \ge 0} (-t^2 \zeta^2)^k / (2k)!$ where $\zeta^2 = \zeta_1^2 + \cdots + \zeta_n^2$. Moreover, $C(t, \zeta) = (e^{itz} + e^{-itz})/2$ if $z \in \mathbb{C}$ has been chosen such that $z^2 = \zeta^2$, and since

$$2(\operatorname{Im} z)^2 = |z^2| - \operatorname{Re}(z^2) \le \sum_{j=1}^n (|\zeta_j^2| - \operatorname{Re}(\zeta_j^2)) = 2 \sum_{j=1}^n (\operatorname{Im} \zeta_j)^2 = 2 |\operatorname{Im} \zeta|^2,$$

one gets the estimate $|C(t, \zeta)| \le e^{|\operatorname{Im} tz|} \le e^{|t|\,|\operatorname{Im} \zeta|}$. The Paley–Wiener theorem (Theorem 1.13) then shows that $C(t, \xi)$ is the Fourier transform of some distribution $c(t)$ with support contained in the ball of radius $|t|$, and since $D_\xi^\alpha C(t, \xi)$ is the Fourier transform of $(-x)^\alpha c(t)$ which also has its support in the same ball, $D_\xi^\alpha C(t, \xi)$ also has a polynomial growth at infinity, i.e.,

$C(t) \in \mathcal{P}$. One can similarly prove the same facts for S, or simply remark that $S(t) = \int_0^t C(s)\,ds$.

The relations $\partial_t C(t,\xi) = -|\xi|^2 S(t,\xi) = (1-\lambda^2(\xi))S(t,\xi)$ and $\partial_t S(t,\xi) = C(t,\xi)$ are immediate and they also imply that $\partial_t^k C(t) \in \mathcal{P}$ and $\partial_t^k S(t) \in \mathcal{P}$ simply by induction. Finally, the estimate $\lambda|S(t)| \leq (1+t^2)^{1/2}$ comes from

$$\lambda^2(\xi)|S(t,\xi)|^2 = \frac{\sin^2 t|\xi|}{|\xi|^2} + \sin^2 t|\xi| \leq t^2 \left(\sup_{s\in\mathbb{R}} \frac{\sin^2 s}{s^2}\right) + 1 = 1 + t^2. \quad \blacksquare$$

The first result we give here is an existence and uniqueness theorem for the Cauchy problem. It also shows that the smoother the data, the smoother the solution.

THEOREM 4.11

Let u_0 and $u_1 \in \mathcal{S}'$ and $f \in C^k(\mathbb{R}; \mathcal{S}')$. Then the Cauchy Problem

$$(\partial_t^2 - \Delta)u(t) = f(t), \qquad u(0) = u_0, \qquad \partial_t u(0) = u_1$$

has a unique solution $u \in C^2(\mathbb{R}; \mathcal{S}')$. Moreover, this solution satisfies

$$\hat{u}(t) = C(t)\hat{u}_0 + S(t)\hat{u}_1 + \int_0^t S(t-s)\hat{f}(s)\,ds,$$

$u \in C^{k+2}(\mathbb{R}; \mathcal{S}')$ and the following:

(i) If u_0 and $u_1 \in \mathcal{S}$ and $f \in C^k(\mathbb{R}; \mathcal{S})$, then $u \in C^{k+2}(\mathbb{R}; \mathcal{S})$.

(ii) If $u_0 \in H^{s+1}$, $u_1 \in H^s$, and $f \in C^0(\mathbb{R}; H^s)$, then $u \in C^0(\mathbb{R}; H^{s+1}) \cap C^1(\mathbb{R}; H^s) \cap C^2(\mathbb{R}; H^{s-1})$.

PROOF Through Fourier transformation in x only, the operator $-\Delta = -\sum_{j=1}^n \partial_j^2$ is transformed into the operator of multiplication by the function $\sum_{j=1}^n \xi_j^2 = |\xi|^2 = \lambda^2(\xi) - 1$. Thus, this Cauchy problem is equivalent to the following:

$$(\partial_t^2 + (\lambda^2 - 1))\hat{u}(t) = \hat{f}(t), \qquad \hat{u}(0) = \hat{u}_0, \qquad \partial_t \hat{u}(0) = \hat{u}_1,$$

where the only variable is t since there is no derivative with respect to the ξ_j's (they are merely parameters).

In view of Lemma 4.10, it is now obvious that the given formula for $\hat{u}(t)$ defines a temperate distribution that solves this problem. For the uniqueness, we have to prove that $\hat{u}(t)$ must be given by this formula. Indeed, if $u \in C^2(\mathbb{R}; \mathcal{S}')$ is a solution of the problem, we can use the distribution $U(s,t) = C(t-s)\hat{u}(s) + S(t-s)\partial_t \hat{u}(s)$ to write

$$\begin{aligned}\hat{u}(t) = U(t,t) &= U(0,t) + \int_0^t \partial_s U(s,t)\,ds \\ &= C(t)\hat{u}_0 + S(t)\hat{u}_1 + \int_0^t S(t-s)\hat{f}(s)\,ds\end{aligned}$$

since $\partial_s U(s,t) = S(t-s)(\partial_t^2\hat{u}(s) + (\lambda^2-1)\hat{u}(s)) = S(t-s)\hat{f}(s)$.

To prove $u \in C^{k+2}(\mathbb{R}; \mathcal{S}')$ we simply compute

$$\partial_t\hat{u}(t) = (1-\lambda^2)S(t)\hat{u}_0 + C(t)\hat{u}_1 + \int_0^t C(t-s)\hat{f}(s)\,ds,$$

$$\partial_t^2\hat{u}(t) = (1-\lambda^2)\hat{u}(t) + \hat{f}(t)$$

and, because $f \in C^k(\mathbb{R}; \mathcal{S}')$, this gives the result.

Finally, the claim (i) is obvious in view of the previous formulas, and for (ii) we remark, using estimates from Lemma 4.10, that the function

$$\begin{aligned}\lambda^{s+1}|\hat{u}(t)| &\le \lambda^{s+1}|C(t)\hat{u}_0| + \lambda^{s+1}|S(t)\hat{u}_1| + \lambda^{s+1}\left|\int_0^t S(t-r)\hat{f}(r)\,dr\right| \\ &\le \lambda^{s+1}|\hat{u}_0| + (1+t^2)^{1/2}\lambda^s|\hat{u}_1| + \left|\int_0^t \lambda S(t-r)\lambda^s\hat{f}(r)\,dr\right|\end{aligned}$$

is square integrable in ξ (for the integral, use the Cauchy–Schwarz inequality), so that $u \in C^0(\mathbb{R}; H^{s+1})$. The proof of $u \in C^1(\mathbb{R}; H^s) \cap C^2(\mathbb{R}; H^{s-1})$ is similar when using the previous formulas for $\partial_t\hat{u}$ and $\partial_t^2\hat{u}$. ∎

When $f = 0$, the same computations lead to the property of conservation of energy.

COROLLARY 4.12

If $u_0 \in H^{s+1}$, $u_1 \in H^s$, and $f = 0$, the energy

$$E_s(t) = \|\partial_t u(t)\|_s^2 + \sum_{j=1}^n \|D_j u(t)\|_s^2$$

of the solution $u \in C^0(\mathbb{R}; H^{s+1}) \cap C^1(\mathbb{R}; H^s)$ obtained in Theorem 4.11(ii) is independent of t and therefore is equal to $E_s = \|u_1\|_s^2 + \sum_{j=1}^n \|D_j u_0\|_s^2$.

PROOF Using the expression of $\hat{u}$ given in Theorem 4.11,

$$\begin{aligned}|\partial_t\hat{u}(t)|^2 + \sum_{j=1}^n |\widehat{D_j u}(t)|^2 &= |(1-\lambda^2)S(t)\hat{u}_0 + C(t)\hat{u}_1|^2 \\ &\quad + (\lambda^2-1)|C(t)\hat{u}_0 + S(t)\hat{u}_1|^2 \\ &= (C(t)^2 + (\lambda^2-1)S(t)^2)(|\hat{u}_1|^2 + (\lambda^2-1)|\hat{u}_0|^2) \\ &= |\hat{u}_1|^2 + (\lambda^2-1)|\hat{u}_0|^2\end{aligned}$$

since $C(t)^2 + (\lambda^2-1)S(t)^2 = \cos^2 t|\xi| + \sin^2 t|\xi| = 1$. The conservation of energy follows by multiplying by λ^{2s} and integrating in ξ. ∎

The last result will describe the phenomenon of finite propagation speed. It shows that a modification of the data cannot immediately affect the values of the solution far from the domain where the data are modified. For the sake of simplicity, we continue to assume $f = 0$.

THEOREM 4.13

Let u_0 and $u_1 \in S'$, $f = 0$ and $u \in C^\infty(\mathbb{R}; S')$ the solution of the Cauchy problem as in Theorem 4.11. Then

(i) If supp $u_0 \cup$ supp u_1 does not intersect the closed ball $\{y \in \mathbb{R}^n; |y-x| \leq |t|\}$, then $(t,x) \notin$ supp u.

(ii) If sing supp $u_0 \cup$ sing supp u_1 does not intersect the sphere $\{y \in \mathbb{R}^n; |y-x| = |t|\}$, then $(t,x) \notin$ sing supp u.

PROOF We will assume $x = 0$ and $t \geq 0$ since the wave operator is invariant under translations in x and reversion of the time.

(i) Let us denote by B_t the closed ball of radius t and by Ω_t the open ball of the same radius; if supp $u_0 \cup$ supp u_1 does not intersect the closed ball B_t, we actually have $u_0 = u_1 = 0$ in some Ω_T for a $T > t$. Then we will prove that for $0 \leq s < T$, $u(s)$ vanishes in Ω_{T-s}, and this will obviously imply that $(t,0) \notin$ supp u.

Thus let $0 \leq s < T$ be fixed and let $\varphi \in C_0^\infty(\Omega_{T-s})$. From Theorem 4.11 we know that $\hat{u}(s) = C(s)\hat{u}_0 + S(s)\hat{u}_1$, and we can write

$$(u(s),\varphi) = (2\pi)^{-n}(\hat{u}(s),\hat{\varphi}) = (2\pi)^{-n}(\hat{u}_0,\overline{C(s)\hat{\varphi}}) + (2\pi)^{-n}(\hat{u}_1,\overline{S(s)\hat{\varphi}}).$$

Actually one has supp $\varphi \subset B_{T-s-\epsilon}$ for some positive ϵ, and thanks to the Paley–Wiener theorem (Theorem 1.13), $\hat{\varphi}$ can be extended as an entire function satisfying estimates $|\hat{\varphi}(\zeta)| \leq C_N(1+|\zeta|^2)^{-N}e^{(T-s-\epsilon)|\text{Im}\,\zeta|}$ for all $N \in \mathbb{Z}_+$ and some sequence of constants C_N. Using the results of Lemma 4.10, we see that the functions $\Phi_0 = C(s)\hat{\varphi}$ and $\Phi_1 = S(s)\hat{\varphi}$ can also be extended as entire functions satisfying estimates

$$|\Phi_j(\zeta)| \leq C'_N(1+|\zeta|^2)^{-N}e^{(T-\epsilon)|\text{Im}\,\zeta|}$$

for all $N \in \mathbb{Z}_+$ and a new sequence C'_N. Again using the Paley–Wiener theorem, it follows that there exist two functions φ_0 and $\varphi_1 \in C_0^\infty$ with supp $\varphi_j \subset B_{T-\epsilon}$, $\hat{\varphi}_0 = C(s)\hat{\varphi}$ and $\hat{\varphi}_1 = S(s)\hat{\varphi}$, so that we can write

$$(u(s),\varphi) = (2\pi)^{-n}(\hat{u}_0,\hat{\varphi}_0) + (2\pi)^{-n}(\hat{u}_1,\hat{\varphi}_1) = (u_0,\varphi_0) + (u_1,\varphi_1),$$

and this is zero since u_0 and u_1 vanish in Ω_T. We thus get $u(s) = 0$ in Ω_{T-s}, as claimed.

(ii) We will first prove that u is smooth in a neighborhood of $\{(s,y) \in \mathbb{R} \times \mathbb{R}^n; s = 0 \text{ and } |y| = t\}$. Indeed, if $|y| = t$, one has $y \notin$ sing supp $u_0 \cup$ sing supp u_1 by assumption, and one can find two functions v_0 and $v_1 \in S$ such that $u_0 = v_0$ and $u_1 = v_1$ in some neighborhood of y. The Cauchy problem with data v_0 and v_1 has a solution $v \in C^\infty(\mathbb{R}; \mathcal{S})$ (cf. Theorem 4.11(i)), and thanks to the part (i) already proved, $(0, y) \notin$ supp $(u - v)$ since $u - v$ is the solution of the Cauchy problem with data $u_0 - v_0$ and $u_1 - v_1$ the supports of which do not intersect $\{y\}$. Therefore u is smooth near $(0, y)$.

To prove that $(t, 0) \notin$ sing supp u, we now use the results of the previous section. If u were singular at $(t, 0)$, there would be a direction $(\tau, \xi) \in \mathbb{R}^{n+1} \setminus 0$ such that $(t, 0; \tau, \xi) \in WFu$ and since $(\partial_t^2 - \Delta)u = 0$, this point $(t, 0; \tau, \xi)$ should lie in the characteristic set of the wave operator by Theorems 4.6 and 4.7. This characteristic set is described by the equation $|\xi|^2 - \tau^2 = 0$ so that we get $\tau = \pm|\xi| \neq 0$. According to the propagation theorem (Theorem 4.8), the whole bicharacteristic curve starting at this point should then lie in WFu. This bicharacteristic curve $(s(r), y(r); \sigma(r), \eta(r))$ is the solution of

$$\frac{ds}{dr} = -2\sigma(r), \quad \frac{dy}{dr} = 2\eta(r), \quad \frac{d\sigma}{dr} = \frac{d\eta}{dr} = 0,$$
$$(s(0), y(0); \sigma(0), \eta(0)) = (t, 0; \tau, \xi)$$

i.e., $(s(r), y(r); \sigma(r), \eta(r)) = (t - 2\tau r, 2\xi r; \tau, \xi)$. For $r = t/2\tau$ we get $s(r) = 0$ while $y(r) = t\xi/\tau$ satisfies $|y(r)| = t|\xi/\tau| = t$, so that this point, which is in sing supp u because of Theorems 4.7 and 4.8, lies on $\{(s, y) \in \mathbb{R} \times \mathbb{R}^n; s = 0$ and $|y| = t\}$ where we proved that u is smooth. This contradiction shows that $(t, 0) \notin$ sing supp u. ∎

A consequence of Theorem 4.13(ii) is that one has the following estimate for the singular support of the solution u:

$$\text{sing supp } u \subset \bigcup_{y \in \sum_0 \cup \sum_1} \{(t, x); |x - y| = |t|\},$$

where $\sum_j =$ sing supp u_j. In other words, the singular support of u is contained in the union of all light cones with vertices located at the singularities of the data. However, the reader will easily understand that the results of Section 4.2 give much more than this estimate.

Indeed, while this inclusion is the most precise one that can be proved when one knows only the singular supports of the data, it is not true that it is an equality. Actually, the singular support of u is much smaller than this union of cones, at least in most of the cases. On the contrary, if we take the point of view of wave front sets, we can give an exact description of the location of singularities of u (and therefore of its singular support). Indeed, denoting the (polyhomogeneous) pseudodifferential operator of symbol $|\xi|$ by $|D|$, it can be proved that $(WFu) \cap \{t = 0\}$ is *equal* to

$$\{(0, x; |\xi|, \xi); (x, \xi) \in WF(u_1 + i|D|u_0)\}$$
$$\cup \{(0, x; -|\xi|, \xi); (x, \xi) \in WF(u_1 - i|D|u_0)\}$$

and then the exact description of WFu follows by using the propagation theorem (Theorem 4.8); cf. Exercise 4.7.

We hope that this simple example will convince the reader of the great usefulness of the microlocal point of view in the study of singularities of solutions of partial differential equations. To close this course, we simply point out that very similar results have been proved for the nonlinear hyperbolic Cauchy problem, and the essential tool in the proof was a well-adapted variant of the theory of pseudodifferential operators. For such results, we refer the reader to Bony [3] and Chemin [5].

Exercises

4.1 *Local solvability of (approximately) invertible operators.* Assume that a and $b \in S^{\infty}$ satisfy $a\#b - 1 \in S^{-\infty}$ (or more generally a and $b \in S^{\infty}_{\rho,0}$ with $a\#b - 1 \in S^{-\infty}_{\rho,0}$, as in Exercises 2.9 and 3.6).

Let $\Omega_\delta = \{x \in \mathbb{R}^n; |x| < \delta\}$ as in Lemma 4.2; show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|\psi\|_{-1} \leq \varepsilon \|\psi\|_0 \text{ for all } \psi \in C_0^{\infty}(\Omega_\delta).$$

(Hint: Show by contradiction that otherwise you could find a square integrable function with support equal to $\{0\}$.)

Show that you can find a $\delta > 0$ and a function $\varphi \in C_0^{\infty}(\Omega_\delta)$ with $\varphi = 1$ near 0 and such that the operator $c(x, D) = \varphi(x)\ (1 - a\#b(x, D))$ satisfies

$$\|c(x, D)\psi\|_{-1} \leq \frac{1}{2}\|\psi\|_{-1} \qquad \text{for all } \psi \in C_0^{\infty}(\Omega_\delta).$$

Let $s \in \mathbb{R}$ and $f \in H^s$; explain why the formulas

$$\psi_0 = c(x, D)f, \qquad \psi_{j+1} = c(x, D)\psi_j, \qquad \text{and } \psi_\infty = \sum_{j \geq 0} \psi_j$$

define functions $\psi_j = C_0^{\infty}(\Omega_\delta)$ (for $j = \infty$, first prove that $\psi_\infty \in H^{-1}$, then observe that $\psi_\infty = \psi_0 + c(x, D)\psi_\infty$). Finally, show that the formula

$$u = b(x, D)(f + \psi_\infty)$$

defines an H^{s-m} distribution solution of $a(x, D)u = f$ in a neighborhood of 0 (here, m denotes the order of b).

4.2 Let $b : \mathbb{R} \to \mathbb{R}$ be a C^∞ function, and let $a(x, \xi) = \xi_1 + ib(x_1)\xi_2$. We want to study the local solvability of $a(x, D)$ at the origin of $\mathbb{R}^2$.

(a) Determine conditions on b equivalent to the following properties:
- $a(x, D)$ is of principal type.
- $a(x, D)$ is principally normal.
- $a(x, D)$ satisfies Hörmander's condition $p = 0 \Rightarrow \{\bar{p}, p\} = 0$, where p is the principal symbol.

Then describe for what functions b the operator $a(x, D)$ is locally solvable (resp. is *not* locally solvable) at the origin thanks to the results given in