In $\mathbb{R}^{n+1}$ we shall use the polar coordinates $x=r\omega$ where $r>0$ and $\omega\in S^n$. Then the Laplace operator assumes the form

$$\Delta=r^{-2}\Delta_\omega+\partial^2/\partial r^2+n/r\,\partial/\partial r$$

outside the origin. Here Δ_ω is the Laplace-Beltrami operator in S^n. If $u(x)=r^\mu v(\omega)$ is a homogeneous function of degree μ, it follows that outside the origin

$$\Delta u=r^{\mu-2}(\Delta_\omega v+\mu(\mu+n-1)\,v).$$

Hence $\Delta u=0$ outside the origin if and only if v is an eigenfunction of $-\Delta_\omega$ with eigenvalue $\lambda=\mu(\mu+n-1)$. Since λ assumes all values $\geqq 0$ when $\mu\leqq 1-n$, we obtain all eigenfunctions of $-\Delta_\omega$ by restricting to S^n all distributions u in $\mathbb{R}^{n+1}$ which are harmonic and homogeneous of degree $\mu\leqq 1-n$ outside the origin. (Cf. Section 3.2.) Then Δu has to be a linear combination of the derivatives of the Dirac measure at 0, so we find that $\mu=1-n-k$ where k is an integer $\geqq 0$, and that

$$u(x)=\sum_{|\alpha|=k} a_\alpha D^\alpha E$$

where E is the fundamental solution of Δ and a_α are constants. (When $k=0$ and $n=1$ the logarithmic potential E must be replaced by a constant. Otherwise there is no harmonic function homogeneous of degree μ in $\mathbb{R}^{n+1}$.) The Fourier transform of u is $\sum a_\alpha \xi^\alpha |\xi|^{-2}$ so it follows that u is supported by the origin if and only if $|\xi|^2$ divides the polynomial $\sum a_\alpha \xi^\alpha$. Let N_k be the dimension of the space of homogeneous polynomials of degree k in $n+1$ variables,

$$N_k=\binom{n+k}{n}=k^n(1+O(1/k))/n!.$$

We define $N_k=0$ for $k<0$. Then it follows that the multiplicity of the eigenvalue $\lambda_k=k(k+n-1)$ of $-\Delta_\omega$ is N_k-N_{k-2} for $k=0,1,\dots$. If $N(\lambda)$ is the number of eigenvalues $\leqq\lambda$ of $-\Delta_\omega$, it follows that

$$N(\lambda_k+0)-N(\lambda_k-0)=N_k-N_{k-2}=2k^{n-1}(1+O(1/k))/(n-1)!.$$

Thus

$$\lambda_k^{(1-n)/2}(N(\lambda_k+0)-N(\lambda_k-0))\to 2/(n-1)!,\qquad k\to\infty,$$

which proves that it is impossible to find an asymptotic formula for $N(\lambda)$ with continuous main term and error $o(\lambda^{(n-1)/2})$.