Problem 21.1. Consider the case of a homogeneous operator $a(D)$ with constant coefficients in $\mathbb{R}^n$. Write down $e(x, y, \lambda)$ as an integral and verify the estimates (21.34), (21.35) and (21.37) directly.

§22. The Laplace Operator on the Sphere

22.1 The Laplace operator on a Riemannian manifold. 1) Let M be a manifold with a Riemannian metric, i.e. in any tangent space $T_x M$ there is given a bilinear form $\langle \cdot, \cdot \rangle$ with a positive definite quadratic form. If $x^1, \ldots, x^n$ are the local coordinates in an open set $U \subset M$, then $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$ is a basis for the tangent space at all points in U. Denoting

$$g_{ij}(x) = \left\langle \left(\frac{\partial}{\partial x^i}\right)_x, \left(\frac{\partial}{\partial x^j}\right)_x \right\rangle, \tag{22.1}$$

we obtain a positive definite matrix $g_{ij}(x)$. If now $v = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j} \in T_x M$, then

$$\langle v, v \rangle = \sum_{i,j=1}^{n} g_{ij}(x) v^i v^j. \tag{22.2}$$

The cotangent space $T_x^* M$ is, by definition, the dual of $T_x M$. A basis of it (for $x \in U$) consists of the 1-forms dx^i, defined by the relations

$$\left\langle dx^i, \frac{\partial}{\partial x^j} \right\rangle = \delta^i_j.$$

A metric on a vector space E induces an isomorphism of E with E^*. With the help of this isomorphism we may transfer the metric from E to E^*. Fixing $x \in U$, let us compute $\langle dx^i, dx^j \rangle$ at x. Which tangent vector corresponds to dx^i? Denote its coordinates by a^{ik}, $k = 1, \ldots, n$; then we have to satisfy the condition

$$\delta^i_j = \left\langle dx^i, \frac{\partial}{\partial x^j} \right\rangle = \left\langle \{a^{ik}\}, \frac{\partial}{\partial x^j} \right\rangle = \sum_{k=1}^{n} g_{kj} a^{ik},$$

from which $a^{ik} = g^{ik}$, the elements of the matrix inverse to $\| g_{ik} \|$. We now have

$$\langle dx^i, dx^j \rangle = \left\langle \sum_{k=1}^{n} g^{ik} \frac{\partial}{\partial x^k}, dx^j \right\rangle = g^{ij}, \tag{22.3}$$

i.e. for any cotangent vector $a = \sum_{i=1}^{n} a_i \, dx^i \in T_x^* M$

$$\langle a, a\rangle = \sum_{i,j=1}^{n} g^{ij} a_i a_j . \tag{22.4}$$

2) Now introduce on the Riemannian manifold a smooth density (volume) such that in the tangent space the volume of a parallelepiped in an orthonormal system equals 1. For an arbitrary parallelepiped, defined by the vectors $e_1, \ldots, e_n$, the volume equals

$$\operatorname{vol}\{e_1, \ldots, e_n\} = |\det\{e_1^{\text{col}}, \ldots, e_n^{\text{col}}\}|, \tag{22.5}$$

where e_i^{col} is the column of coordinates of e_i in the orthonormal basis. Formula (22.5) follows from the fact that the volume has to be an additive, non-negative invariant of parallelepipeds.

What about the case where the coordinate basis is not orthonormal? Let us then consider the operator A, mapping an orthonormal basis $e'_1, \ldots, e'_n$ into the basis $e_1, \ldots, e_n$. The columns of its matrix (in the basis $e'_1, \ldots, e'_n$), will be the coordinates of the vectors e_i in the orthonormal basis e'_i, so that $\operatorname{vol}\{e_1, \ldots, e_n\} = |\det A| = \sqrt{\det(A^*A)}$.

Now, the matrix elements of A^*A in the basis $\{e'_i\}$ are of the form

$$\langle A^*Ae'_i, e'_j\rangle = \langle Ae'_i, Ae'_j\rangle = \langle e_i, e_j\rangle . \tag{22.6}$$

Therefore, the volume of the parallelepiped $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$ equals $\sqrt{g}$, where $g = \det\|g_{ij}(x)\|$ and the volume of an arbitrary set F in the local coordinates of U is defined as

$$\operatorname{vol}(F) = \int_F \sqrt{g(x)}\, dx^1 \ldots dx^n . \tag{22.7}$$

It follows directly from the change of variable formula for an integral that the integral (22.7) is independent of the choice of local coordinates. We may therefore take it immediately as a definition of $\operatorname{vol}(F)$, defining a smooth positive density dv on M by the formula

$$dv = \sqrt{g(x)}\, dx^1 \ldots dx^n . \tag{22.8}$$

3) For any differential operator

$$A\colon C^\infty(M) \to C^\infty(M)$$

there exists a unique operator A^*, such that

$$(Af, g) = (f, A^*g), \quad f, \ g \in C_0^\infty(M), \tag{22.9}$$

where

$$(f, g) = \int f(x)\overline{g(x)}\, dv. \tag{22.10}$$

Analogous arguments hold also for operators on sections of vector bundles over M, if in each fiber of the bundle we have a hermitean metric (positive definite hermitean form), which replaces $f(x)\overline{g(x)}$ in (22.10).

Let us define a scalar product on the space of 1-forms $\Lambda^1(M)$, taking on $T_x^*M \otimes \mathbb{C}$ the hermitean scalar product induced by the metric on T_x^*M introduced above. Consider the operator

$$d\colon\ C^\infty(M) \to \Lambda^1(M), \tag{22.11}$$

mapping a function $f \in C^\infty(M)$ to its differential $df \in \Lambda^1(M)$, defined by the fact that if $v \in T_xM$, then $\langle df, v\rangle = (vf)(x)$, where $(vf)(x)$ denotes the derivative of f in the direction v. In coordinates:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\, dx^i.$$

The operator $\delta\colon \Lambda^1(M) \to C^\infty(M)$ is defined as the adjoint of d, i.e. $\delta = d^*$.

Definition 22.1. The *Laplace* (*or Laplace-Beltrami*) *operator* on functions

$$\Delta\colon\ C^\infty(M) \to C^\infty(M)$$

on a Riemannian manifold M, is defined by the formula

$$\Delta = -\delta \cdot d. \tag{22.12}$$

Analogously, the Laplace operator is defined on p-forms $\Lambda^p(M)$ as

$$\Delta = -(d\delta + \delta d)\colon\ \Lambda^p(M) \to \Lambda^p(M),$$

but we will not consider this case.

It is immediately clear that the Laplace operator has the following properties:

a) $\Delta^* = \Delta$;

b) if $T\colon M \to M$ preserves the metric on the tangent spaces and $\hat{T}\colon C^\infty(M) \to C^\infty(M)$ is defined by $\hat{T}f = f \circ T$, then

$$\Delta \hat{T} = \hat{T} \Delta,$$

i.e. Δ commutes with isometries;

c) if M is closed, then $(\Delta f, f) \leq 0$ and from $\Delta f = 0$ it follows that $f = \text{const}$.

4) Let us compute δ and Δ in local coordinates. We have

$$\left(\delta\left(\sum_{i=1}^{n} a_i dx^i\right), f\right) = \left(\sum_{i=1}^{n} a_i dx^i, \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j\right)$$

$$= \sum_{i,j=1}^{n} \int g^{ij} a_i \frac{\partial \bar{f}}{\partial x^j} \sqrt{g}\, dx = \sum_{i,j=1}^{n} \int \bar{f}\left[-\frac{\partial}{\partial x^j}(\sqrt{g}\, g^{ij} a_i)\right] dx$$

$$= \int \bar{f} \cdot \sum_{i,j} \frac{1}{\sqrt{g}}\left[-\frac{\partial}{\partial x^j}(\sqrt{g}\, g^{ij} a_i)\right] \sqrt{g}\, dx,$$

from which

$$\delta\left(\sum_{i=1}^{n} a_i dx^i\right) = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(\sqrt{g}\, g^{ij} a_i) \tag{22.13}$$

and

$$\Delta f = \delta\, df = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^i}\right). \tag{22.14}$$

Example 22.1. In the Euclidean space $\mathbb{R}^n$ with its standard metric, we obtain

$$\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^{i2}}.$$

Note that all invariants of Δ are also invariants of the Riemannian manifold, in particular, the residues and the values of the ζ-function.

Exercise 22.1. Compute in local coordinates the principal symbol of the Laplace operator on a Riemannian manifold.

22.2 The Laplace operator on the sphere S^n. The n-sphere S^n is the following submanifold of $\mathbb{R}^{n+1}$:

$$S^n = \left\{(x^0, \ldots, x^n) : \sum_{i=0}^{n} (x^i)^2 = 1\right\}.$$

There is a metric on S^n, induced by the standard metric on $\mathbb{R}^{n+1}$ and correspondingly a Laplace operator, which we denote by Δ_S. There is the following method for computing Δ_S, using the operator Δ on $\mathbb{R}^{n+1}$.

Proposition 22.1. *Let $f(\omega)$ be a function on S^n and extend it to $\mathbb{R}^{n+1}$, putting*

$$\hat{f}(x) = f\left(\frac{x}{|x|}\right)$$

i.e. by homogeneity of degree 0. *Then*

$$\Delta_S f = \Delta \hat{f}\,|_{S^n}, \tag{22.15}$$

or

$$(\Delta \hat{f})(x) = r^{-2} \Delta_S f, \tag{22.16}$$

where $r = |x|$.

Proof. The equivalence of (22.15) and (22.16) is obvious, since $\Delta \hat{f}$ has degree of homogeneity -2.

Let us prove (22.15). Denote temporarily by Δ_S' the operator on S^n, defined by the right-hand side of (22.15).

The group of isometries $SO(n+1)$ acts on S^n by restricting to S^n the rotations of $\mathbb{R}^{n+1}$. Using this group, any unit tangent vector may be mapped into any other such vector. Therefore, the principal symbol of an operator commuting with all isometries, is constant on all unit cotangent vectors (and, consequently, is uniquely defined up to a scalar multiplier). Clearly the operators Δ_S and Δ_S' commute with the action of $SO(n+1)$ on functions. Therefore, there exists a constant λ such that the operator $\Delta_S' - \lambda \Delta_S$ has order 1. But this operator also commutes with $SO(n+1)$ and since there are no linear functions which are invariant with respect to rotations and since the multiplication term of $\Delta_S' - \lambda \Delta_S$ is constant because of rotational symmetry, then the operator $\Delta_S' - \lambda \Delta_S$ is a multiple of the identity, and, consequently, is zero, because $\Delta_S' 1 = \Delta_S 1 = 0$. Hence $\Delta_S' = \lambda \Delta_S$. From what follows, it will be clear that $\lambda = 1$, on the other hand this can be verified by computing Δ_S and Δ_S' on any non-harmonic function on the sphere. □

22.3 Eigenvalues of the operator Δ_S. Let us compute Δ in polar coordinates. Let $r = |x|$ and $\omega = x/|x|$, then

$$\Delta(f(r)\,g(\omega)) = (\Delta f)\,g + f(\Delta g) + 2(\nabla f)\cdot(\nabla g) = (\Delta f) g + f(\Delta g), \tag{22.17}$$

since $(\nabla f)\cdot(\nabla g) = 0$ (∇f is directed along the radius vector and ∇g is directed along the tangent). Let us compute $\Delta f(r)$. We have

$$r'_{x^i} = \frac{x^i}{r}, \quad r''_{x^i x^i} = \frac{1}{r} - \frac{(x^i)^2}{r^3}.$$

From this we obtain

$$\Delta f(r) = \sum_{i=0}^{n} \frac{\partial}{\partial x^i}\left(\frac{x^i}{r} f'(r)\right) = \sum_{i=0}^{n} f''(r)\,\frac{(x^i)^2}{r^2}$$

$$+ \sum_{i=0}^{n}\left(\frac{1}{r} - \frac{(x^i)^2}{r^3}\right) f'(r) = f''(r) + \frac{n}{r} f'(r).$$

Therefore

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S, \tag{22.18}$$

where Δ_S is applied on the unit sphere with subsequent extension by homogeneity of degree zero. Formula (22.18) is true also on arbitrary functions (linear combinations of functions of the type $f(r)g(\omega)$ are dense in the space $C^\infty(\mathbb{R}^{n+1}\setminus 0)$).

In particular, for $f(r)=r^\mu$, we obtain

$$\begin{aligned}\Delta(r^\mu g(\omega)) &= r^{-2+\mu}[\Delta_S g + [\mu(\mu-1)+n\mu]g]\\ &= r^{-2+\mu}[\Delta_S g + \mu(\mu+n-1)g].\end{aligned} \tag{22.19}$$

It follows from (22.19), that the following proposition holds

Proposition 22.2. *The equality $\Delta(r^\mu g(\omega))=0$ (for $r\neq 0$) is equivalent to the fact that $g(\omega)$ is an eigenfunction of the operator $-\Delta_S$ with eigenvalue $\lambda=\mu(\mu+n-1)$.*

Since all eigenvalues of the operator $-\Delta_S$ are non-negative, we may assume that $\mu\geqq 0$ or $\mu\leqq 1-n$. Note that the quadratic function $\lambda(\mu)=\mu(\mu+n-1)$ takes all values $\lambda\geqq 0$ for $\mu\geqq 0$ and each of them exactly once. It is therefore clear that the eigenvalues $\lambda\geqq 0$ are in a one-to-one correspondence with those $\mu\geqq 0$, for which there exists a non-trivial function $g(\omega)$ such that $r^\mu g(\omega)$ is a harmonic function on $R^{n+1}\setminus 0$. But then, by the removable singularity theorem, the function $r^\mu g(\omega)$ is harmonic everywhere on $\mathbb{R}^{n+1}$ and, consequently, is a harmonic polynomial by the Liouville theorem. In particular, μ is an integer. Clearly the converse is also true, i.e. the restrictions to S^n of homogeneous, harmonic polynomials are eigenfunctions of the operator $-\Delta_S$ with eigenvalues $\lambda=k(k+n-1)$, where $k=0,1,2,\dots$ and by the maximum principle we see that a harmonic polynomial is uniquely defined by its restriction to S^n. Hence, we have proven

Theorem 22.1. *The eigenvalues of the operator $-\Delta_S$ are $\lambda=k(k+n-1)$, where $k=0,1,2,\dots$ and the multiplicity of the eigenvalue $k(k+n-1)$ equals the dimension of the space of homogenous, harmonic polynomials of degree k.*

22.4 Computing the multiplicity. Let M_k be the space of homogeneous polynomials of degree k. Let us compute $N_k=\dim M_k$. A basis in M_k is given by the monomials $x_0^{k_0}\dots x_n^{k_n}$, where $k_0+k_1+\dots+k_n=k$. The number of ordered partitions of k into a sum of $n+1$ non-negative numbers equals the number of ordered partitions of $k+n+1$ into a sum of $n+1$ positive numbers, which in turn equals the number of ways of choosing n from $n+k$, i.e. equals

$$N_k=\binom{n+k}{n}=\frac{(n+k)!}{n!\,k!}=\frac{1}{n!}(k+n)(k+n-1)\dots(k+1). \tag{22.20}$$

Note that Δ determines a map

$$\Delta\colon M_k \to M_{k-2} \tag{22.21}$$

and one has the exact sequence

$$0 \to H_k \to M_k \to M_{k-2} \tag{22.22}$$

where $H_k = \operatorname{Ker} \Delta|_{M_k}$ is the space of homogeneous, harmonic polynomials of degree k, the dimension of which we would like to compute.

Theorem 22.2. 1) *The operator $\Delta\colon M_k \to M_{k-2}$ is surjective.*

2) *There is a direct sum decomposition*

$$M_k = \bigoplus_{k-2l \geq 0} r^{2l} H_{k-2l}, \tag{22.23}$$

where $r^2 = x_0^2 + \ldots + x_n^2$.

Corollary 22.1. a)

$$\dim H_k = N_k - N_{k-2}. \tag{22.24}$$

b) *If f and φ are two polynomials, then there exists a unique polynomial u such that*

$$\Delta u = f, \quad u|_{|x|=1} = \varphi|_{|x|=1} \tag{22.25}$$

(*i.e. the Dirichlet problem for the Poisson equation in the unit ball is solvable in polynomials*).

c) *The harmonic polynomials cannot be divided by r^2.*

Derivation of Corollary 22.1 from Theorem 22.2. a) The relation (22.24) follows from the fact that the sequence (22.22) can be rewritten in the form

$$0 \to H_k \to M_k \to M_{k-2} \to 0. \tag{22.26}$$

b) In view of part 1) of the theorem, the solvability of (22.25) reduces to the case $f = 0$, where it is obvious in view of (22.23).

c) Also obvious in view of (22.23). □

Proof of Theorem 22.2. Both statements are proved at the same time by induction in k.

For $k = 0, 1$ the statements of the theorem are true. Let them be true also for all $l < k$.

1°. Show that $H_k \cap r^2 M_{k-2} = 0$. Since by the inductive hypothesis we have

$$M_{k-2} = \bigoplus_{k-2l-2 \geq 0} r^{2l} H_{k-2l-2}$$

then

$$r^2 M_{k-2} = \bigoplus_{\substack{k-2l \geq 0 \\ l>0}} r^{2l} H_{k-2l}$$

and if $h_k \in H_k \cap r^2 M_{k-2}$ then

$$h_k = \sum_{\substack{l>0 \\ k-2l \geq 0}} c_l r^{2l} h_{k-2l}. \tag{22.27}$$

Now consider the harmonic polynomial $h = h_k - \sum_{l>0} c_l h_{k-2l}$. We have $\deg h \leq k$ and h_k is the homogeneous component of h of degree k. Since $h|_{|x|=1} = 0$, then $h \equiv 0$ from which $h_k \equiv 0$, as required.

2°. From the condition $M_k \supset H_k \oplus r^2 M_{k-2}$, we obtain

$$\dim H_k \leqq N_k - N_{k-2} \tag{22.28}$$

where the equality is equivalent to the decomposition

$$M_k = H_k \oplus r^2 M_{k-2} \tag{22.29}$$

3°. From the exact sequence (22.22) it follows that

$$\dim H_k = N_k - \dim \Delta(M_k) \geqq N_k - N_{k-2}, \tag{22.30}$$

where the equality is equivalent to the surjectivity of the operator

$$\Delta\colon M_k \to M_{k-2}$$

4°. From (22.28) and (22.30) it follows that $\dim H_k = N_k - N_{k-2}$ from which follows the surjectivity and the decomposition (22.29) and hence, by the inductive hypothesis the decomposition (22.23) also follows. □

Corollary 22.2. *The multiplicity of the eigenvalue $\lambda = k(k+n-1)$ of the operator $-\Delta_S$ is equal to $N_k - N_{k-2}$, where N_k is given by the formula* (22.20).

22.5 The function $N(\lambda)$ for the operator $-\Delta_S$. It is clear that

$$N(k(k+n-1)+0) = \sum_{l \leq k} \dim H_l = \sum_{l \leq k} (N_l - N_{l-2}) = N_k + N_{k-1}. \tag{22.31}$$

But from (22.20) it is also clear that N_k is a polynomial of degree n in k with leading coefficient $1/n!$. It therefore follows from (22.31) that

$$N(k(k+n-1)+0) \sim \frac{2}{n!} k^n \sim \frac{2}{n!} [k(k+n-1)]^{n/2}. \tag{22.32}$$

From (22.31) it also follows that

$$N(k(k+n-1)+0) - N(k(k+n-1)-0) = N_k - N_{k-2} = P_{n-1}(k),$$

where $P_{n-1}(k)$ is a polynomial in k of degree $n-1$. Therefore

$$N(k(k+n-1)+0) - N(k(k+n-1)-0) \geqq ck^{n-1} \geqq c[k(k+n-1)]^{\frac{n-1}{2}}, \quad (22.33)$$

where $c>0$. From (22.32) and (22.33) it is clear that

$$N(\lambda) = \frac{2}{n!}\lambda^{n/2}(1+O(\lambda^{-1/2})), \quad (22.34)$$

where the term $O(\lambda^{-1/2})$ is sometimes greater than $c\lambda^{-1/2}$, i.e. the remainder estimate is the best possible (in any case as far as the exponent is concerned).

Let us verify that (22.34) is exactly the asymptotic formula from Theorem 21.2 (which, in particular, also shows that the multiplier λ from the proof of Proposition 22.1 equals 1). We have to verify that

$$2/n! = (2\pi)^{-n} V_n V_n' \quad (22.35)$$

where V_n is the volume of the unit n-ball and V_n' is the area of S^n (Riemannian volume).

Clearly $V_n = \frac{V_{n-1}'}{n}$ and from this everything reduces to the identity

$$V_{n-1}' V_n' = \frac{2(2\pi)^n}{(n-1)!} \quad (22.36)$$

Let us compute V_{n-1}' and prove this relation. Put $I = \int\limits_0^\infty e^{-x^2}dx$; then

$$I^n = \int\limits_{x_i \geqq 0} e^{-(x_1^2+\dots+x_n^2)}dx_1 \dots dx_n$$

and in particular

$$I^2 = \int\limits_{x_i \geqq 0} e^{-(x_1^2+x_2^2)}dx_1\,dx_2 = \int\limits_0^{\pi/2} d\varphi \int\limits_0^\infty re^{-r^2}dr = \frac{\pi}{4}\int\limits_0^\infty e^{-z}dz = \frac{\pi}{4},$$

from which $I = \frac{\sqrt{\pi}}{2}$, $I^n = \frac{\pi^{n/2}}{2^n}$. Now note that

$$I^n = \frac{V_{n-1}'}{2^n}\int\limits_0^\infty r^{n-1}e^{-r^2}dr = \frac{V_{n-1}'}{2^n}\cdot\frac{1}{2}\int\limits_0^\infty z^{\frac{n}{2}-1}e^{-z}dz = \frac{V_{n-1}'}{2^{n+1}}\Gamma\left(\frac{n}{2}\right),$$

from which we obtain $V'_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

As is well-known, integration by parts yields $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for $\alpha > 0$, from which for $n = 2k$ we obtain

$$\Gamma\left(\frac{n}{2}\right) = \Gamma(k) = (k-1)! = \left(\frac{n}{2}-1\right)!.$$

If now $n = 2k+1$, then

$$\Gamma\left(\frac{n}{2}\right) = \Gamma\left(k+\frac{1}{2}\right) = \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right),$$

but $\Gamma\left(\frac{1}{2}\right) = \frac{2\sqrt{\pi}}{V'_0} = \sqrt{\pi}$, from which

$$\Gamma\left(\frac{n}{2}\right) = \frac{n-2}{2}\cdot\frac{n-4}{2}\cdots\frac{1}{2}\sqrt{\pi} = 2^{-\frac{n-1}{2}}(n-2)!!\,\sqrt{\pi}.$$

Now let n be even. Then

$$\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right) = \left(\frac{n}{2}-1\right)!\,2^{-n/2}(n-1)!!\,\sqrt{\pi}$$

$$= 2^{-\left(\frac{n}{2}-1\right)}(n-2)!!\,2^{-n/2}(n-1)!!\,\sqrt{\pi} = 2\cdot 2^{-n}(n-1)!\,\sqrt{\pi},$$

from which

$$V'_{n-1}V'_n = \frac{2\pi^{n/2}\,2\cdot\pi^{(n+1)/2}}{2\cdot 2^{-n}(n-1)!\,\sqrt{\pi}} = \frac{2(2\pi)^n}{(n-1)!},$$

as required. The case of odd n is considered in the same way.

Problem 22.1. Write down the expression for the Laplace operator in the Lobacevskii plane (hyperbolic plane).

Problem 22.2. Compute the eigenvalues for the Laplace operator on the n-dimensional real projective space $\mathbb{R}P^n$ with the natural metric (induced by the metric on the sphere S^n under the two-fold covering $S^n \to \mathbb{R}P^n$).

Problem 22.3. Compute the eigenvalues of the Laplace operator on the n-dimensional complex projective space $\mathbb{C}P^n$ with the natural Kähler metric (cf. S.S. Chern [1]).