4.1 Local solvability of linear differential operators

Consider a linear differential operator $a(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ with complex-valued C^∞ coefficients a_α. Remember that its principal symbol $p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is then a complex-valued C^∞ function on $T^*\mathbb{R}^n$. We want to study the following property: the operator $a(x, D)$ is said to be *locally solvable at* x_0 if there exists a neighborhood Ω of x_0 such that $a(x, D)u = f$ has a solution $u \in \mathcal{D}'(\Omega)$ for any $f \in C_0^\infty(\Omega)$.

It can be proved that this property is equivalent to some a priori estimate[1] for the operator $a^*(x, D)$. The next problem then is to translate this a priori estimate into a checkable geometric condition on the symbol a. To state such a condition, we introduce the Poisson bracket $\{p, q\}$ of two complex-valued C^1 functions on $T^*\mathbb{R}^n$ defined as

$$\{p,q\}(x,\xi) = \langle \partial_\xi p(x,\xi), \partial_x q(x,\xi)\rangle - \langle \partial_x p(x,\xi), \partial_\xi q(x,\xi)\rangle.$$

This quantity appears naturally as the principal symbol (up to a factor i) of the commutator $[a(x, D), b(x, D)] = (a\#b - b\#a)(x, D)$ when p and q are the principal symbols of $a(x, D)$ and $b(x, D)$ (cf. the asymptotic formula in Theorem 2.7). In [7, Chap.6], Hörmander proved that the principal symbol p of a locally solvable operator $a(x, D)$ must satisfy $\{\bar{p}, p\} = 0$ on $p = 0$, but the proof is too long and technical to be rewritten here. In this section, we will just prove a converse of this result, the statement of which requires the following definitions.

The operator $a(x, D)$ is said to be of *principal type at* x_0 if the ξ-gradient of its principal symbol at x_0 vanishes only for $\xi = 0$. It is said to be *principally normal at* x_0 if there exists a function $q \in C^\infty(T^*\mathbb{R}^n \setminus 0)$ homogeneous of degree $m - 1$ in ξ such that the principal symbol p satisfies

$$\{\bar{p}, p\}(x,\xi) = 2i\mathrm{Re}\,(\bar{q}(x,\xi)p(x,\xi)) \quad \text{for} \quad \xi \in \mathbb{R}^n \setminus 0 \text{ and } x \text{ close to } x_0.$$

REMARK In order to prove that $a(x, D)$ is of principal type at x_0, it is sufficient to check that $\partial_\xi p(x_0,\xi) \neq 0$ *when* $p(x_0, \xi) = 0$ in view of Euler's identity $p(x,\xi) = (1/m)\langle \partial_\xi p(x,\xi), \xi\rangle$. In order to prove that $a(x, D)$ is principally normal at x_0, it is also sufficient to check that property *near the zeroes of* p. Indeed, one can always write $\{\bar{p}, p\} = 2i\mathrm{Re}\,(\bar{q}p)$ near (x_0, ξ_0) if $p(x_0, \xi_0) \neq 0$ because it suffices to take $q = \{\bar{p}, p\}/2i\bar{p}$ as long as $p \neq 0$. Thus, if one can also write $\{\bar{p}, p\} = 2i\mathrm{Re}\,(\bar{q}p)$ near any zero of p, the compact set $K = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n;\ x = x_0 \text{ and } |\xi| = 1\}$ can be covered by open sets Ω_j where one has $\{\bar{p}, p\} = 2i\mathrm{Re}\,(\bar{q}_j p)$. Then, using the partition of unity constructed

[1] The Latin words *a priori* mean "before"; an *a priori estimate for an operator A* is an inequality between a norm of $A\varphi$ and a norm of φ proved for a whole class of functions φ (typically for all $\varphi \in C_0^\infty(\Omega)$ for some Ω) *before* assuming anything on φ; it is only eventually that it will be used for special φ's.

in Lemma 1.5, the function $q_0 = \sum \varphi_j q_j$ satisfies $\{\bar{p}, p\} = 2i\mathrm{Re}\,(\bar{q}_0 p)$ in a neighborhood of K, which finally gives $\{\bar{p}, p\} = 2i\mathrm{Re}\,(\bar{q}p)$ for $\xi \in \mathbb{R}^n \setminus 0$ and x close to x_0 by homogeneity if we set $q(x, \xi) = |\xi|^{m-1} q_0(x, \xi/|\xi|)$. Principally normal operators obviously satisfy Hörmander's condition $\{\bar{p}, p\} = 0$ on $p = 0$. The converse will be discussed at the end of this section (see Corollary 4.4 and its comments). ▮

The main result of this section is the following.

THEOREM 4.1
Let $a(x, D)$ be an mth *order principally normal operator of principal type at x_0. Then there exists a neighborhood Ω of x_0 such that the equation $a(x, D)u = f$ (in Ω) has a solution $u \in L^2(\Omega)$ for any $f \in H^{1-m}$.*

Example
Operators with real or constant coefficients in the principal part satisfy $\{\bar{p}, p\} \equiv 0$, and therefore they are principally normal (take $q = 0$). Elliptic operators have nonvanishing principal symbols (except at $\xi = 0$), so that — thanks to the previous remark — they satisfy the assumptions of Theorem 4.1. The student will find other operators satisfying these assumptions by solving the following questions. If a is of principal type, is a^* also of principal type? If a is principally normal, is a^* also principally normal? Is $a\#b$ of principal type when a and b are of principal type? What about the converse? Assuming that $a\#b$ is of principal type, that a and b are principally normal, and that the order of c is smaller than the order of $a\#b$, does $a\#b + c$ satisfy the assumptions of Theorem 4.1? ▯

In the proof of Theorem 4.1, our first step will be to show that the principal type and the normality correspond to some a priori estimates, but before stating and proving this result, we give a lemma that provides estimates for functions with small support. In the following, we denote by $\Omega_\delta = \{x \in \mathbb{R}^n; |x| < \delta\}$ the open ball of radius $\delta > 0$.

LEMMA 4.2
For all $\delta > 0$ and $m \in \mathbb{Z}_+$,

$$\|\varphi\|_m \le 2\delta \|\varphi\|_{m+1} \qquad \text{for all } \varphi \in C_0^\infty(\Omega_\delta).$$

Moreover, if Q and R are differential operators of orders m and $2m$ respectively, there exists a constant C such that for all $\varphi \in C_0^\infty(\Omega_\delta)$

$$\|Q(ix_j\varphi)\|_0 \le C\delta \|\varphi\|_m \qquad \text{and} \qquad |(ix_j\varphi, R\varphi)| \le C\delta \|\varphi\|_m^2.$$

PROOF The first inequality is proved by induction. Indeed, it is sufficient to prove it for $m = 0$ in view of Proposition 1.14, and since $\|D_1\varphi\|_0 \le \|\varphi\|_1$ one can write

$$\|\varphi\|_0^2 = (\varphi, \varphi) = (D_1(ix_1\varphi), \varphi) - (ix_1(D_1\varphi), \varphi)$$
$$= (ix_1\varphi, D_1\varphi) + (D_1\varphi, ix_1\varphi) \le 2\|ix_1\varphi\|_0\|D_1\varphi\|_0 \le 2\delta\|\varphi\|_0\|\varphi\|_1.$$

For the second inequality, let us write $Q(ix_j\varphi) = [Q, ix_j]\varphi + ix_j(Q\varphi)$, which gives

$$\|Q(ix_j\varphi)\|_0 \le \|[Q, ix_j]\varphi\|_0 + \|ix_j(Q\varphi)\|_0 \le C\|\varphi\|_{m-1} + C\delta\|\varphi\|_m$$

since $[Q, ix_j]$ has order $m - 1$, and we get the result from the first inequality.

Finally, for the third inequality, we write R as a sum of differential monomials of order $2m$, i.e., $R = \sum_k Q_k Q'_k$ for some mth order operators Q_k and Q'_k, and the result comes from the second inequality since

$$|(ix_j\varphi, R\varphi)| = \left|\sum_k (Q_k^*(ix_j\varphi), Q'_k\varphi)\right| \le \sum_k \|Q_k^*(ix_j\varphi)\|_0\|Q'_k\varphi\|_0 \quad \blacksquare$$

PROPOSITION 4.3

Let $a(x, D)$ be a linear differential operator of order m. Then,

(i) If $a(x, D)$ is of principal type at 0, there exist a $\delta_0 > 0$ and a C_0 such that for all $\delta < \delta_0$ and $\varphi \in C_0^\infty(\Omega_\delta)$,

$$\|\varphi\|_{m-1}^2 \le C_0\delta(\|a(x, D)\varphi\|_0^2 + \|a^*(x, D)\varphi\|_0^2 + \|\varphi\|_{m-1}^2).$$

(ii) If $a(x, D)$ is principally normal at 0, there exist a $\delta > 0$ and a C such that for all $\varphi \in C_0^\infty(\Omega_\delta)$,

$$\|a(x, D)\varphi\|_0^2 \le C(\|a^*(x, D)\varphi\|_0^2 + \|\varphi\|_{m-1}^2).$$

(iii) If $a(x, D)$ is both principally normal and of principal type at 0, there exists a $\delta > 0$ such that for all $\varphi \in C_0^\infty(\Omega_\delta)$,

$$\|\varphi\|_{m-1} \le \|a^*(x, D)\varphi\|_0.$$

PROOF (i) Let $A = a(x, D)$, $Q_j = [A, ix_j] = (\partial_{\xi_j} a)(x, D)$, $B = \sum_{j=1}^n Q_j^* Q_j = b(x, D)$. Thanks to Theorem 2.7, $b = \sum_{j=1}^n |\partial_{\xi_j} p|^2$ modulo S^{2m-3}, and since A is of principal type we have by homogeneity $\sum_{j=1}^n |\partial_{\xi_j} p(x, \xi)|^2 \ge 2\epsilon|\xi|^{2m-2}$ for some $\epsilon > 0$ and all x in some $\Omega_{2\delta_0}$. Therefore, the symbol $b + \epsilon\lambda^{2m-2}\#(1 - \psi)$ satisfies the assumption of Theorem 3.9 (Gårding's inequality) if $\psi \in C_0^\infty(\Omega_{2\delta_0})$ and $\psi \le 1$. Moreover, if $\psi = 1$ in Ω_{δ_0} and $\delta < \delta_0$, $(1 - \psi)\varphi = 0$ for all

$\varphi \in C_0^\infty(\Omega_\delta)$ then $(b(x,D) + \epsilon\lambda^{2m-2}(D)(1-\psi))\varphi = B\varphi$. It follows that we have

$$2\sum_{j=1}^{n} \|Q_j\varphi\|_0^2 = 2\mathrm{Re}\,(B\varphi,\varphi) \geq \epsilon\|\varphi\|_{m-1}^2 - C\|\varphi\|_{m-2}^2$$

for some constant C.

On the other hand, for each operator Q_j one can write

$$\begin{aligned}\|Q_j\varphi\|_0^2 &= (A(ix_j\varphi) - ix_j(A\varphi), Q_j\varphi)\\ &= (ix_j\varphi, A^*Q_j\varphi) - (ix_j(A\varphi), Q_j\varphi)\\ &= (ix_j\varphi, [A^*, Q_j]\varphi) + (Q_j^*(ix_j\varphi), A^*\varphi) - (ix_j(A\varphi), Q_j\varphi).\end{aligned}$$

For $\varphi \in C_0^\infty(\Omega_\delta)$, this can be estimated by using Lemma 4.2:

$$\begin{aligned}\|Q_j\varphi\|_0^2 &\leq C_{j,1}\delta\|\varphi\|_{m-1}^2 + C_{j,2}\delta\|\varphi\|_{m-1}\|A^*\varphi\|_0 + C_{j,3}\delta\|A\varphi\|_0\|\varphi\|_{m-1}\\ &\leq C_j\delta(\|A\varphi\|_0^2 + \|A^*\varphi\|_0^2 + \|\varphi\|_{m-1}^2),\end{aligned}$$

and since the same Lemma 4.2 also implies $\|\varphi\|_{m-2}^2 \leq 4\delta^2\|\varphi\|_{m-1}^2$, this gives

$$\|\varphi\|_{m-1}^2 \leq \frac{2}{\epsilon}\sum_{j=1}^{n}\|Q_j\varphi\|_0^2 + \frac{C}{\epsilon}\|\varphi\|_{m-2}^2 \leq C_0\delta(\|A\varphi\|_0^2 + \|A^*\varphi\|_0^2 + \|\varphi\|_{m-1}^2)$$

as required.

(ii) Let us modify the function q near $\xi = 0$ so that q is now C^∞ everywhere while the relation $\{\bar{p}, p\} = 2i\mathrm{Re}\,(\bar{q}p)$ holds only for $|\xi| \geq 1$ and x in some $\Omega_{2\delta}$. Then for $\psi \in C_0^\infty(\Omega_{2\delta})$ we set

$$b = \psi a \in S^m, \qquad c = \psi q + i\{a,\psi\} \in S^{m-1}$$

and

$$r = b^*\#b - b\#b^* - b\#c^* - c\#b^* \in S^{2m}.$$

Actually, the symbol r is in S^{2m-2} because by using the asymptotic formulas of Theorem 2.7, we can write modulo S^{2m-2}: $b^* = \bar{b} - i\bar{b}_{\langle x,\xi\rangle}$ (where we use the notation $a_{\langle x,\xi\rangle} = \sum_j \partial_{x_j}\partial_{\xi_j}a$; similarly, we will use the notation $a_x = \partial_x a$ and $a_\xi = \partial_\xi a$), then

$$\begin{aligned}r &= (\bar{b} - i\bar{b}_{\langle x,\xi\rangle})b - i\langle\bar{b}_\xi, b_x\rangle - b(\bar{b} - i\bar{b}_{\langle x,\xi\rangle}) + i\langle b_\xi, \bar{b}_x\rangle - b\bar{c} - c\bar{b}\\ &= -i\{\bar{b}, b\} - 2\mathrm{Re}\,(\bar{c}b) = -i(\{\bar{b}, b\} - 2i\mathrm{Re}\,(\bar{c}b))\\ &= -i\psi^2(\{\bar{a}, a\} - 2i\mathrm{Re}\,(\bar{q}a)) = -i\psi^2(\{\bar{p}, p\} - 2i\mathrm{Re}\,(\bar{q}p))\end{aligned}$$

and this is identically zero for $|\xi| \geq 1$.

Therefore, if we take ψ such that $\psi = 1$ in Ω_δ and we write A, B, Q, and R for $a(x, D)$, $b(x, D)$, $c(x, D)$, and $r(x, D)$, then $B\varphi = A\varphi$ and $B^*\varphi = A^*\varphi$ for all $\varphi \in C_0^\infty(\Omega_\delta)$ since A has the local property, and we can write

$$\begin{aligned}\|A\varphi\|_0^2 &= (B^*B\varphi, \varphi) = (R\varphi, \varphi) + (BB^*\varphi, \varphi) + (BQ^*\varphi, \varphi) + (QB^*\varphi, \varphi)\\ &= (R\varphi, \varphi) + \|A^*\varphi\|_0^2 + 2\mathrm{Re}\,(Q^*\varphi, A^*\varphi)\\ &\le \|R\varphi\|_{1-m}\|\varphi\|_{m-1} + 2\|A^*\varphi\|_0^2 + \|Q^*\varphi\|_0^2\\ &\le 2\|A^*\varphi\|_0^2 + C\|\varphi\|_{m-1}^2\end{aligned}$$

as required, since $R \in \Psi^{2m-2}$ and $Q^* \in \Psi^{m-1}$.

(iii) Finally, when both assumptions hold, we get from (i) and (ii) that for small $\delta > 0$ and for $\varphi \in C_0^\infty(\Omega_\delta)$,

$$\|\varphi\|_{m-1}^2 \le C_1\delta(\|a^*(x, D)\varphi\|_0^2 + \|\varphi\|_{m-1}^2)$$

for a new constant C_1, and if we now choose $\delta < 1/2C_1$, one has

$$\begin{aligned}\|\varphi\|_{m-1}^2 &= 2\|\varphi\|_{m-1}^2 - \|\varphi\|_{m-1}^2\\ &\le 2C_1\delta(\|a^*(x, D)\varphi\|_0^2 + \|\varphi\|_{m-1}^2) - \|\varphi\|_{m-1}^2 \le \|a^*(x, D)\varphi\|_0^2. \quad \blacksquare\end{aligned}$$

PROOF OF THEOREM 4.1 The property of local solvability now follows from the a priori estimate thanks to a classic argument from functional analysis. Indeed, using a translation, we can assume that $x_0 = 0$, and we take $\delta > 0$ as in Proposition 4.3(iii). Then, the operator $a^*(x, D)$ is injective on $C_0^\infty(\Omega_\delta)$, and we can consider its inverse $(A^*)^{-1}$ which is well defined on the space

$$\mathbf{E} = \{\psi \in C_0^\infty(\Omega_\delta); \exists\varphi \in C_0^\infty(\Omega_\delta) \text{ with } \psi = a^*(x, D)\varphi\}.$$

For $f \in H^{1-m}$ we define on $\mathbf{E}$ the semi-linear form $U(\psi) = (f, (A^*)^{-1}\psi)$ which satisfies

$$|U(\psi)| = |(f, \varphi)| \le \|f\|_{1-m}\|\varphi\|_{m-1} \le \|f\|_{1-m}\|a^*(x, D)\varphi\|_0 = \|f\|_{1-m}\|\psi\|_0$$

where we used the estimate (iii) of Proposition 4.3 for $\varphi = (A^*)^{-1}\psi$. Thus, U is continuous for the L^2 norm. We can then extend this form as a continuous form on $L^2(\Omega_\delta)$ (by the Hahn–Banach theorem, or in a more elementary way by the following Hilbertian arguments: by continuity to the closure $\mathbb{F}$ of $\mathbf{E}$, then by setting $U(\psi) = U(\pi_{\mathbb{F}}\psi)$ where $\pi_{\mathbb{F}}$ denotes the Hilbertian projection on the closed subspace $\mathbb{F}$), and by using Riesz's representation theorem we get the existence of a $u \in L^2(\Omega_\delta)$ such that $(u, \psi) = U(\psi)$ for $\psi \in \mathbf{E}$, i.e.,

$$(u, a^*(x, D)\varphi) = (f, \varphi) \qquad \text{for all } \varphi \in C_0^\infty(\Omega_\delta),$$

which means $a(x, D)u = f$ in Ω_δ by definition. $\blacksquare$

To close this section, we finally compare Hörmander's result quoted at the beginning of this section with the result of Theorem 4.1.

The first situation we consider is when the sets $\text{Re}\, p = 0$ and $\text{Im}\, p = 0$ intersect transversally. Technically, this leads to the following transversality assumption: $p(x_0, \xi) = 0$ and $\xi \neq 0 \Rightarrow \text{Re}\, \partial_\xi p(x_0, \xi)$ and $\text{Im}\, \partial_\xi p(x_0, \xi)$ are linearly independent. Then we have:

COROLLARY 4.4

Let $a(x, D)$ be a linear differential operator with principal symbol p satisfying the following statement: the real and imaginary parts of the ξ-gradient of p are linearly independent at (x_0, ξ) for all solutions $\xi \neq 0$ of $p(x_0, \xi) = 0$. Then $a(x, D)$ of principal type at x_0, and the following three conditions are equivalent:

- *(i)* *$a(x, D)$ is principally normal at x_0.*
- *(ii)* *$a(x, D)$ is locally solvable at x_0.*
- *(iii)* *$a(x, D)$ satisfies Hörmander's condition $\{\bar{p}, p\} = 0$ on $p = 0$ in a neighborhood of x_0.*

PROOF Under the transversality assumption, $a(x, D)$ is obviously of principal type at x_0 in view of the remark following the definitions given above.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow respectively from Theorem 4.1 and from Hörmander [7, Theorem 6.1.1], so that we only need to prove (iii) $\Rightarrow$ (i) under the transversality assumption. Again, thanks to the remarks following the definitions given above, it is sufficient to prove that one can write $\{\bar{p}, p\} = 2i\text{Re}\,(\bar{q}p)$ *near the zeroes of* p; at such points, the transversality assumption shows that $\text{Re}\, p$ and $\text{Im}\, p$ can be taken as local coordinates in $\mathbb{R}^{2n}$ and a Taylor formula thus gives

$$\frac{1}{2i}\{\bar{p}, p\} = \frac{1}{2i}\{\bar{p}, p\}|_{p=0} + q_1 \text{Re}\, p + q_2 \text{Im}\, p$$

for some coefficients q_1 and $q_2 \in C^\infty(\mathbb{R}^{2n})$. Finally, Hörmander's condition (iii) allows to rewrite this relation $\{\bar{p}, p\} = 2i\text{Re}\,(\bar{q}p)$ by setting $q = q_1 + iq_2$. ∎

The next step would be to weaken the transversality assumption in that corollary, and a natural substitute is to require that the ξ-gradient of p never vanishes at zeroes of p, in other words, to assume that $a(x, D)$ is an operator of principal type. In this situation, Hörmander's condition is no longer equivalent to the normality of $a(x, D)$. As a matter of fact, it can even be proved that Hörmander's condition is not sufficient and the normality is not necessary for the local solvability. However, a more precise characterization of local solvability is known for these operators of principal type (the so-called condition (P)), but the proof is much more difficult and requires the introduction of much wider classes of pseudodifferential operators. Indeed, this result, mainly due to Nirenberg and