J.P. Solovej: Lieb-Thirring and Cwikel-Lieb-Rozenblum inequalities

Schrödinger operators on $\mathbb{R}^d$: $-\Delta+V$ with semiclassical parameter $-h^2\Delta+V=H_h$, $h\downarrow 0$, $V:\mathbb{R}^d\to\mathbb{R}$ measurable. Will give conditions on V s.t. H_h is bounded below on $C_0^\infty(\mathbb{R}^d)$, i.e. $\forall \varphi\in C_0^\infty(\mathbb{R}^d)$: $\langle \varphi, H_h\varphi\rangle_{L^2}\geq -C_h\|\varphi\|^2_{L^2}$ for some $C_h\in\mathbb{R}$.

Minimal assumption on V: $V\in L^2_{loc}(\mathbb{R}^d)$.

For operators bounded from below we have the Friedrichs extension. This is the one we will consider.

Weyl's law: Let
$$g_s(t)=\left\{\begin{array}{ll} \mathbb{1}_{(-\infty,0)}(t), & s=0\\ |t|\,\mathbb{1}_{(-\infty,0)}(t), & s=1\end{array}\right\} = |t|^s\,\mathbb{1}_{(-\infty,0)}(t).$$

We would expect (under appropriate conditions) Weyl's law
$$\mathrm{Tr}\; g_s(H_h) = (2\pi)^{-d}\iint_{\mathbb{R}^d\,\mathbb{R}^d} g_s(h^2p^2+V(x))\,dp\,dx + o(h^{-d}) \quad \text{as } h\to 0$$
$$= (2\pi h)^{-d}\iint g_s(p^2+V(x))\,dp\,dx + o(h^{-d})$$

Theorem: a) For all $d\in\mathbb{N}$ if $V\in L^{\frac{d+2}{2}}(\mathbb{R}^d)$ then $\forall h\in(0,1]$,

(Lieb-Thirring)
$$\mathrm{Tr}\; g_1(H_h) \leq C_d(2\pi h)^{-d}\iint g_1(p^2+V(x))\,dp\,dx$$
$$= C_d \underbrace{\frac{2(2\pi h)^{-d}}{d+2}\,K_d}_{=:C^1_{cl}}\int |V_-(x)|^{\frac{d+2}{2}}\,dx$$
K_d: volume of unit ball in $\mathbb{R}^d$

Lieb-Thirring conjecture: $C_d=1$ is ok.

b) For all $d\geq 3$, $h>0$, $V\in L^{d/2}(\mathbb{R}^d)$ then

(CLR)
$$\mathrm{Tr}\; g_0(H_h)\leq C_d^0\,(2\pi h)^{-d}\iint g_0(p^2+V(x))$$
$$= C_d\, C^0_{cl}\int |V_-(x)|^{d/2}\,dx$$

Known: $C_d^0>1$.

We will prove (a) with $C_d^1=\left(\frac{d+4}{d}\right)^{d/2}$.

Note that: $g_1(t) = \int_0^{\infty} g_0(t+t')\,dt' = \int_0^{|t|} 1\,dt' = |t|$ (if $t<0$).

This allows to deduce the $s=1$-case from the $s=0$-case.

We will prove that $\varphi_1, \dots, \varphi_N \in C_0^\infty(\mathbb{R}^d)$ orthonormal in $L^2(\mathbb{R}^d)$, then

$$(*) \quad \sum_{j=1}^{N} (\varphi_j, (-h^2\Delta + V)\varphi_j)_{L^2(\mathbb{R}^d)} \geq -C_d (2\pi h)^{-d} \iint g_1(p^2+V(x))\,dp\,dx$$

$N=1 \Rightarrow$ lower bound for $-h^2\Delta + V$,

arbitrary N: $\Rightarrow$ no essential spectrum below 0.

<u>Proof of $(*)$</u>: $e>0$, Let $h=1$. $\varphi_j^{e,\pm}$ defined by

$$\hat{\varphi}_j^{e,+}(p) = \begin{cases} \hat{\varphi}_j(p), & p^2 > e \\ 0, & p^2 < e \end{cases}$$

$$\hat{\varphi}_j^{e,-}(p) = \begin{cases} 0, & p^2 > e \\ \hat{\varphi}_j(p), & p^2 < e \end{cases}$$

$\Rightarrow \varphi_j = \varphi_j^{e,+} + \varphi_j^{e,-}$, and

① $$\int_0^\infty de \int dx\, |\varphi_j^{e,+}(x)|^2 = \int_0^\infty de \int_{p^2>e} dp\, |\hat{\varphi}_j^{e,+}(p)|^2 = \int dp\, |\hat{\varphi}_j(p)|^2 \int_0^{p^2} de\, 1 = \int p^2 |\hat{\varphi}_j(p)|^2\,dp = \int |\nabla\varphi_j(x)|^2 = (\varphi_j, -\Delta\varphi_j)$$

② Δ-inequality in $\mathbb{C}^N$

$$\left(\sum_{j=1}^N |\varphi_j^{e,+}(x)|^2\right)^{1/2} \geq \left[\left(\sum_{j=1}^N |\varphi_j(x)|^2\right)^{1/2} - \left(\sum_{j=1}^N |\varphi_j^{e,-}(x)|^2\right)^{1/2}\right]_+$$

③ $$\sum_{j=1}^N |\varphi_j^{e,-}(x)|^2 = \sum_{j=1}^N \left| (2\pi)^{-d/2} \int e^{ipx}\, \mathbb{1}_{\{p^2<e\}}(p)\, \hat{\varphi}_j(p)\,dp \right|^2$$

$$\underset{\{\varphi_j\} \text{ orthonormal}}{\leq} (2\pi)^{-d} \int |e^{ipx}\, \mathbb{1}_{\{p^2<e\}}(p)|^2\,dp = K_d\,(2\pi)^{-d}\, e^{d/2}$$

independently of N!

④ $$\sum_{j=1}^N (\varphi_j, -\Delta\varphi_j) \geq \int_{\mathbb{R}^d} \left(\int_0^\infty \left[\left(\sum_{j=1}^N |\varphi_j(x)|^2\right)^{1/2} - \left((2\pi)^{-d} K_d\, e^{d/2}\right)^{1/2}\right]_+^2 de \right) dx$$

As $\displaystyle\int_0^\infty \left[A - B e^{d/4}\right]_+^2 \, de = \int_0^{(A/B)^{4/d}} \left(A - B e^{d/4}\right)^2 de = \text{const } A^2 \left(\frac{A}{B}\right)^{4/d}$

we get
$$\sum_{j=1}^N (\varphi_j, -\Delta \varphi_j) \geq \int_{\mathbb{R}^d} \frac{(2\pi)^2 d^2 K_d^{-2/d}}{(d+2)(d+4)} \left(\sum_{j=1}^N |\varphi_j(x)|^2\right)^{\frac{d+2}{2}} dx .$$

(5)
$$\sum_{j=1}^N (\varphi_j, (-h^2\Delta + V)\varphi_j) \geq \int_{\mathbb{R}^d} \frac{(2\pi h)^2 d^2 K_d^{-2/d}}{(d+2)(d+4)} \left(\sum_{j=1}^N |\varphi_j(x)|^2\right)^{\frac{d+2}{d}} dx + \int_{\mathbb{R}^d} \underbrace{V(x)}_{\geq -|V_-(x)|} \sum_{j=1}^N |\varphi_j(x)|^2 \, dx$$

Hölder
$$\int_{\mathbb{R}^d} V(x) \sum_{j=1}^N |\varphi_j(x)|^2 \, dx \geq -\left(\int \left(\sum_{j=1}^N |\varphi_j(x)|^2\right)^{\frac{d+2}{d}} dx\right)^{d/d+2} \left(\int |V_-|^{\frac{d+2}{2}} dx\right)^{2/d+2}$$

$$\geq \frac{(2\pi h)^2 d^2 K_d^{-2/d}}{(d+2)(d+4)} X - \left(\int |V_-|^{\frac{d+2}{2}} dx\right)^{2/d+2} X^{d/d+2}, \quad X = \int_{\mathbb{R}^d} \left(\sum_{j=1}^N |\varphi_j(x)|^2\right)^{\frac{d+2}{d}}$$

Minimize in X:
$$\geq - \frac{2(2\pi h)^{-d} K_d}{d+2} \left(\frac{d+4}{d}\right)^{d/2} \int |V_-(x)|^{\frac{d+2}{2}} dx \qquad \square$$

<u>CLR:</u> Same method, but use the following trick:

Choose $\left\{\sqrt{-h^2\Delta}\, \varphi_j\right\}_{j=1}^N$ orthonormal.

One can also use this to prove Weyl's law. A major ingredient is that $g_0(A+B) \leq g_0(A) + g_0(B)$ "Ky Fan inequality".