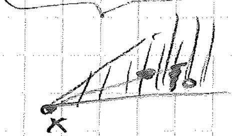
Wave front sets

Lemma: $v \in \mathcal{E}'$. Then $v \in C_c^\infty \iff |\hat{v}(\xi)| \leq C_N (1+|\xi|)^{-N} \; \forall N$

Def:
- Let $\Omega \subseteq \mathbb{R}^n$ open, $u \in \mathcal{D}'(\Omega)$. u is C^∞ at $x \in \Omega$ if $\exists \varphi \in C_c^\infty : \varphi \equiv 1$ near x s.t. $\varphi u \in C_c^\infty$.
- $\operatorname{sing\,supp}(u)$ is the set of points $x \in \Omega$ where u is not C^∞.

Def:
- u is C^∞ at x in a direction $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ if $\exists v \in \mathcal{E}'$ s.t. $u = v$ near x and $|\hat{v}(\xi)| \leq C_N (1+|\xi|)^{-N}$ for ξ s.t. $\underbrace{\left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \varepsilon}$ for some $\varepsilon > 0$.



cone around ξ_0 of opening angle $\sim \varepsilon$

- $WF(u) = \{ (x_0, \xi_0) \in \Omega \times \mathbb{R}^n \setminus \{0\} : u \text{ is not } C^\infty \text{ at } x_0 \text{ in direction } \xi_0 \}$

Example: a) $\delta_{(x,y)} \in \mathcal{D}'(\mathbb{R}^2)$, $\langle \delta_{(x,y)}, \varphi \rangle = \varphi(0,0)$

$\operatorname{supp} \delta_{(x,y)} = \{(0,0)\}$

$\Rightarrow \delta_{(x,y)}$ C^∞ outside $(x,y) = (0,0)$ in all directions

Let $\varphi \in C_c^\infty$, $\varphi \equiv 1$ near $(0,0)$.

$|\mathcal{F}(\underbrace{\varphi\, \delta_{(x,y)}}_{v})| = |\varphi(0,0)| = 1$ does not decay rapidly

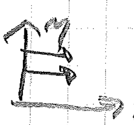
$\Rightarrow WF(\delta_{(x,y)}) = \{ (0,0,\xi,\eta) : (\xi,\eta) \in \mathbb{R}^2 \setminus \{0\} \}$

b) $\delta_x \in \mathcal{D}'(\mathbb{R}^{n-k} \times \mathbb{R}^k)$, $\langle \delta_x, \varphi \rangle = \int \varphi(0,y)\, dy$

$\operatorname{supp} \delta_x = \{x = 0\} \Rightarrow \delta_x$ is C^∞ outside $x = 0$.

$\varphi \in C_c^\infty$, $\varphi \equiv 1$ near $(0,y)$,

$|\mathcal{F}(\varphi \delta_x)| = |\mathcal{F}(\varphi(0,y)\delta_x)| = |(\mathcal{F}_{y \to \eta} \varphi) 1_\xi|$

$\Rightarrow$  δ_x not smooth in ξ-direction

Show that it is smooth in any other direction (i.e. for $\eta \neq 0$)

$|\mathcal{F}(\varphi \delta_x)| = |\underbrace{\mathcal{F}_{y \to \eta} \varphi}_{\in S(\mathbb{R}^k)} 1_\xi| \leq C_N (1+|\eta|)^{-N} \; \forall N$

Need $(1+|\eta|)^{-N} \le C\,(1+|\xi,\eta|)^{-N}$, true for directions $\neq (\xi,0)$.

$\Rightarrow \quad WF(\delta_X) = \{(0,y,\xi,0) : y \in \mathbb{R}^k,\ \xi \in \mathbb{R}^{n-k}\setminus\{0\}\}$

Lemma: If $\varphi \in C_c^\infty$, $(x_0,\xi_0) \notin WF(u) \Rightarrow (x_0,\xi_0) \notin WF(\varphi u)$

Proof: Assume that $\hat{v}$ rapidly decreasing in a cone Γ around (x_0,ξ_0)

Let $R \ge 0$.

$$|\widehat{\varphi v}(\xi)| \le \int_{|\eta|\le R} |\hat{v}(\xi-\eta)|\,|\hat{\varphi}(\eta)|\,d\eta + \int_{|\eta|>R} \dots$$

$$\le C_R \sup_{|\eta|\le R} |\hat{v}(\xi-\eta)| + C_L \int_{|\eta|>R} |\hat{v}(\xi-\eta)|\,(1+|\eta|)^{-L}\,d\eta \quad \forall L$$

Diff Fun 2: $\mathcal{F}\mathcal{E}'_R = \{ f \in \mathcal{O}(\mathbb{C}^n) : \exists\, C, N : |f(z)| \le C e^{R|\mathrm{Im}\, z|}(1+|z|)^N \ \forall z \}$

$$\le C_R \sup|\dots| + \tilde{C}_L \int_{|\eta|>R} (1+|\xi-\eta|)^N (1+|\eta|)^{-L}\,d\eta$$

$$\le \dots + \tilde{C}_L (1+|\xi|)^N \int_{|\eta|>R} (1+|\eta|)^{N-L}\,d\eta$$

$$\le \dots + \tilde{C}_{L,R} (1+|\xi|)^N R^{N-L+n}$$

Need control over $|\hat{v}(\xi-\eta)|$. Take $R = |\xi|^{1/2}$ and $\Gamma' \subset \Gamma$ with $\varepsilon' < \varepsilon$

If $\xi \in \Gamma' \Rightarrow \xi-\eta \in \Gamma$ when $|\xi|$ is large and $|\eta| \le R$.

Note that
$$\left|\frac{(\xi-\eta)}{|\xi-\eta|} - \frac{\xi_0}{|\xi_0|}\right| \le \left| \frac{\xi}{|\xi|} \overbrace{\left(\frac{|\xi|}{|\xi-\eta|} - 1\right)}^{\in[-2\tilde{\varepsilon},\,2\tilde{\varepsilon}]} - \frac{\eta}{|\xi|}\,\frac{|\xi|}{|\xi-\eta|} \right| + \underbrace{\left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right|}_{<\varepsilon'} \quad \text{for } R \ge \tilde{\varepsilon}^{-1}$$

Using the triangle inequality several times: $\le 3\tilde{\varepsilon} + 2\tilde{\varepsilon}^2 + \varepsilon' < \varepsilon$

if $\tilde{\varepsilon}$ is small ($\le \frac{\varepsilon-\varepsilon'}{5}$).

$$\Rightarrow |(\widehat{\varphi v})(\xi)| \leq C_{R,\tilde{N}}(1+|\xi-\eta|)^{-\tilde{N}} + \tilde{C}_{LR}(1+|\xi|)^{N} R^{N-L-n}$$

As $1-R^{-1} \leq \frac{|\xi-\eta|}{|\xi|} \leq 1+R^{-1}$, $|\xi-\eta| \sim |\xi|$

$$\leq C_N(1+|\xi|)^{-N}$$ for $|\xi|$ large in a smaller cone.

Also bounded for small ξ $\Rightarrow$ rapidly decreasing $\forall \xi$. $\square$

(Note that the C_R, $\tilde{C}_R$ can actually be chosen independent of R, because $\varphi \in C_c^\infty$).

<u>Cor:</u> In the def. of $WF(u)$, one can take $v = \varphi u$ for some $\varphi \in C_c^\infty$ which is $\equiv 1$ near x.

<u>Lemma:</u> Let $\pi: \Omega \times \mathbb{R}^n \setminus 0 \to \Omega$, $(x,\xi) \mapsto x$, $u \in \mathcal{D}'(\Omega) \Rightarrow \pi\, WF(u) = \text{sing supp}\, u$.

See Shubin, App A for the simple proofs.

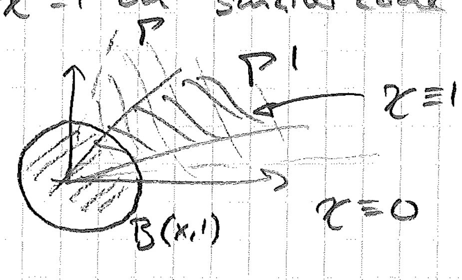
<u>Prop:</u> For $u \in \mathcal{D}'$, $(x_0,\xi_0) \notin WF(u) \Rightarrow \exists$ classical ΨDO^0 A s.t.

1.) A properly supported

2.) The symbol $a \equiv 1$ mod $S^{-\infty}$ in a conic neighborhood of (x_0,ξ_0)

3.) $Au \in C_c^\infty$

<u>Proof:</u> Let $\varphi \in C_c^\infty$, $\varphi \equiv 1$ near x s.t. $\widehat{\varphi u}$ rapidly decreasing in a cone Γ around (x_0,ξ_0). Let $\chi \in C_c^\infty$, supp $\chi \subseteq \Gamma$, $\chi \equiv 1$ on smaller cone $\Gamma' \subset \Gamma$ s.t. $\chi(t\xi) = \chi(\xi)$ $\forall t, |\xi| \geq 1$



Then $|\chi(\xi)\,\widehat{\varphi u}(\xi)| \le \left(\sup_{\xi\in B(x,1)} |\chi(\xi)|\right) |\widehat{\varphi u}(\xi)|$

DifFun2: $\varphi u \in \mathcal{E}' \Rightarrow \widehat{\varphi u}(\xi) = \langle \varphi u, e^{-ix\xi}\rangle$ continuous as a composition of continuous functions

$\Rightarrow \chi\,\widehat{\varphi u} \in C(\mathbb{R}^n) \cap \{\text{rapidly decreasing}\}$

$\Rightarrow \xi^\alpha \chi \widehat{\varphi u}(\xi) \in L^1 \quad \forall \alpha$

$\Rightarrow |D^\alpha \mathcal{F}^{-1}(\chi\,\widehat{\varphi u})| = |\mathcal{F}^{-1}(\underbrace{\xi^\alpha \chi\,\widehat{\varphi u}(\xi)}_{\in L^1})| \in C(\mathbb{R}^n)\ \forall\alpha$

$\Rightarrow \mathcal{F}^{-1}(\chi\,\widehat{\varphi u}) \in C^\infty$

Let $\psi \in C_c^\infty$, $\psi \equiv 1$ near x, $\psi(x)\,\mathcal{F}^{-1}(\chi\,\widehat{\varphi u}) \in C_c^\infty$

Claim: $A = \psi(x)\,\chi(D)\,\varphi(x)$ satisfies the requirements

0.) A classical of order 0, because $|D^\alpha \chi| \le C_\alpha \langle\xi\rangle^{-|\alpha|}\ \forall\alpha$ and ψ, φ are DO of order 0.

1.) easily verified from definition

2.) $a \sim \underbrace{\psi(x)\chi(\xi)\varphi(x)}_{\substack{\equiv 1 \text{ because} \\ \psi,\varphi\equiv 1 \text{ near } x}} + \underbrace{\text{terms with (derivatives of } \chi\text{)}\cdot\text{(derivatives of } \varphi\text{)}}_{=0\text{, because } \chi\equiv 1 \text{ in } \Gamma'}$

3.) was verified above $\square$

Example: Heaviside fct $H(x) =$ 

Consider $u = \mathcal{F}^{-1}H$

Lemma: For $\varphi \in C_c^\infty$ $\forall N$: $|\widehat{\varphi u}(\xi)| \le C_N (1+|\xi|)^{-N}$ for $\xi < 0$

$|\widehat{\varphi u}(\xi) + 2\pi\varphi(0)| \le C_N (1+|\xi|)^{-N}$ for $\xi > 0$

Assuming this, for (x_0, ξ_0), $\varphi \equiv 1$ near x_0, $\varphi(0) = 0$ if $x_0 \ne 0$.

If $x_0 \ne 0 \Rightarrow \widehat{\varphi u}$ rapidly decreasing $\Rightarrow (x_0,\xi_0) \notin WF(u)$

If $x_0 = 0 \Rightarrow \widehat{\varphi u}$ rapidly decreasing for $\xi < 0$, but

$\widehat{\varphi u} \xrightarrow{|\xi|\to\infty} 2\pi$ for $\xi > 0 \Rightarrow$ not rapidly decreasing

$\Rightarrow WF(u) = \{(0,\xi) : \xi > 0\}$.

1 Examples of wave front sets

Let $H : \mathbb{R} \to \mathbb{R}$ be the Heaviside function, $H(x) = 1$ if $x \geqslant 0$ and $H(x) = 0$ otherwise, and consider the distribution $u = \mathcal{F}^{-1}H$. We are going to show that $WF(u) = \{0\} \times (0, \infty)$.

Lemma 1. *Let $\phi \in C_c^\infty(\mathbb{R})$. Then for all N: $|\mathcal{F}(\phi u)(\xi)| \leq C_N(1 + |\xi|)^{-N}$ for $\xi < 0$ and $|\mathcal{F}(\phi u)(\xi) + (2\pi)^{\cdots}\phi(0)| \leq C_N(1 + |\xi|)^{-N}$ for $\xi > 0$.*

Proof.

$$\mathcal{F}(\phi u) = \mathcal{F}(\phi \mathcal{F}^{-1}H) = (\mathcal{F}\phi) * \mathcal{F}(\mathcal{F}^{-1}H) = (\mathcal{F}\phi) * H.$$

As $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R})$, the right had side is the convolution of two functions and

$$\mathcal{F}(\phi u)(\xi) = \int_{\mathbb{R}} (\mathcal{F}\phi)(\xi - \eta)H(\eta)\ d\eta = \int_0^\infty (\mathcal{F}\phi)(\xi - \eta)d\eta = \int_\xi^{-\infty} (\mathcal{F}\phi)(\eta)d\eta.$$

For $\xi < 0$, we can use that $\mathcal{F}\phi$ is rapidly decreasing, to estimate

$$\left|\int_\xi^{-\infty} (\mathcal{F}\phi)(\eta)d\eta\right| \leq C_N \int_{-\infty}^{\xi} (1+|\eta|)^{-N-2}d\eta \leq C_N(1+|\xi|)^{-N} \int_{-\infty}^{0} (1+|\eta|)^{-2}d\eta\ ,$$

so $|\mathcal{F}(\phi u)(\xi)| \leq \tilde{C}_N(1 + |\xi|)^{-N}$. On the other hand

$$-(2\pi)^{\cdots}\phi(0) = -(2\pi)^{\cdots}\mathcal{F}^{-1}(\mathcal{F}\phi)(0) = \int_\infty^{-\infty} e^{i0\eta}(\mathcal{F}\phi)(\eta)d\eta = \int_\infty^{-\infty} (\mathcal{F}\phi)(\eta)d\eta\ ,$$

and

$$\int_\xi^{-\infty} (\mathcal{F}\phi)(\eta)d\eta + (2\pi)^{\cdots}\phi(0) = \int_\xi^{-\infty} (\mathcal{F}\phi)(\eta)d\eta - \int_\infty^{-\infty} (\mathcal{F}\phi)(\eta)d\eta = \int_\xi^{\infty} (\mathcal{F}\phi)(\eta)d\eta\ .$$

Analogous to the above argument we obtain

$$\left|\int_\xi^{-\infty} (\mathcal{F}\phi)(\eta)d\eta + (2\pi)^{\cdots}\phi(0)\right| \leq \int_\xi^{\infty} |(\mathcal{F}\phi)(\eta)|d\eta \leq C_N(1+|\xi|)^{-N} \int_0^{\infty} (1+|\eta|)^{-2}d\eta$$

for $\xi > 0$. □

Therefore, given (x_0, ξ_0), let ϕ be any test function which is identically 1 in a neighborhood of x_0 and, if $x_0 \neq 0$, vanishes in 0. By the Lemma, $|\mathcal{F}(\phi u)(\xi)|$ is rapidly decreasing unless $x_0 = 0$ and $\xi \in (0, \infty) = \mathbb{R}_+\xi_0$. Therefore $WF(u) = \{0\} \times (0, \infty)$. Note that in one dimension the conical ε-neighborhoods around ξ_0 are just one of the half-axes $(0, \pm\infty)$.

Sabrina is also going to discuss the distribution $\delta_x \in \mathcal{D}'(\mathbb{R}_x^{n-k} \times \mathbb{R}_y^k)$ defined by $\langle \delta_x, \phi \rangle := \int_{\mathbb{R}^k} \phi(0, y)\ dy$. δ_x is the usual delta distribution if $k = 0$, and $\delta_x = 1$ if $k = n$. In general, δ_x is a distribution supported on {x=0}. One readily computes $WF(\delta_x) = \{((0, y), (\xi, 0))\ :\ y \in \mathbb{R}^k,\ 0 \neq \xi \in \mathbb{R}^{n-k}\}$ and, for $f \in C^\infty(\mathbb{R}^n)$, $WF(f\delta_x) = \{((0, y), (\xi, 0))\ :\ (0, y) \in \text{supp} f\ ,\ 0 \neq \xi \in \mathbb{R}^{n-k}\}$.

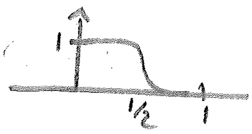
Microlocal parametrix:

$=:$ a elliptic near (x_0,ξ_0)

Let $a \in S^m$, a_m the hom. extension of the princ. symbol, $a_m(x_0,\xi_0) \neq 0$.

$\overset{\text{continuity}}{\Longrightarrow}$ $\exists$ neighborhoods U of x_0, $\varepsilon > 0$ s.t. $a_m(x,\xi) \neq 0$ $\forall x \in U$ $\forall |\xi - \xi_0| < \varepsilon$

$\overset{\text{homogeneity}}{\Longrightarrow}$ $\exists \tilde{\varepsilon}: a_m(x,\xi) \neq 0 \quad \forall x \in U \quad \forall \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \tilde{\varepsilon}$.

Let $\chi \in C_c^\infty(U)$, $\chi \equiv 1$ near x_0, $\tilde{\chi} =$ 

$$b_{-m} := a_m^{-1}(x,\xi) \underbrace{\chi(x)\, \tilde{\chi}\left(\tfrac{1}{\varepsilon}\left|\tfrac{\xi}{|\xi|} - \tfrac{\xi_0}{|\xi_0|}\right|\right) (1 - \tilde{\chi}(|\xi|))}_{=: \psi_\varepsilon(x,\xi) \in S^0 \;=:\; \chi(x)\varphi_\varepsilon(\xi)} \in S^{-m}$$

$b_{-m} \# a =: \psi_\varepsilon - r$, $r \in S^{-1}$ supported on supp b_{-m}.

$r' \sim \sum_{n \geq 0} r^{\# n}$, $\underbrace{(r' \# b_{-m})}_{b} \# a(x,\xi) \sim 1$ in a conical neighborhood of (x_0,ξ_0).

Conclusion: $\exists$ B properly supported ΨDO^{-m} s.t. symbol of $B = b$, symbol of $B\,op(a) = 1$ in a conical neighborhood of (x_0,ξ_0).

Prop. A.1.2: $u \in \mathcal{D}'(X)$, $(x_0,\xi_0) \in X \times (\mathbb{R}^n \setminus \{0\})$, $A = op(a) \in S^m$ prop. supported (or $u \in \mathcal{E}'$), $a_m(x_0,\xi_0) \neq 0$, $Au \in C^\infty$

$\Longrightarrow (x_0,\xi_0) \notin WF(u)$

Proof: a) Multiplying by B from above $\Rightarrow$ $BA \in S^0$ prop supp. if A prop supp., $BAu \in C^\infty$, $\sigma(BA)(x,\xi) \sim 1$ near (x_0,ξ_0)

So may assume $a(x,\xi) \sim 1$ in conical neighborhood of (x_0,ξ_0)

$\Rightarrow$ for ε small: $\chi(x)\varphi_\varepsilon(\xi)\, a(x,\xi) \sim \chi(x)\varphi_\varepsilon(\xi)$

Composition formula: $\varphi_\varepsilon(D)\chi(x) A \sim \underbrace{\varphi_\varepsilon(D)\chi(x)}^{=op(c)} op(a)$

$$\sim op\Big(\sum_\alpha b_\alpha \partial^\alpha c\, \partial^\alpha_\xi a\Big) = op(c)$$

$a \equiv 1$ on on supp c

$$= \varphi_\varepsilon(D)\chi(x)$$

$$\Rightarrow \quad \varphi_\varepsilon(D)\chi(x)u = \varphi_\varepsilon(D)\chi(x)\underbrace{Au}_{\in C^\infty} + \underbrace{Ru}_{\in C^\infty} \in C^\infty$$

$R \in op(S^{-\infty})$

c) Show: $\varphi_\varepsilon(D)\chi(x)u \in S(\mathbb{R}^n)$

(Then $\varphi_\varepsilon(\xi)\widehat{\chi u}(\xi) \in S(\mathbb{R}^n)$
$\Rightarrow \widehat{\chi u}(\xi)$ rapidly $\downarrow$ on $\{\varphi_\varepsilon(\xi) \equiv 1\}$.)

The claim follows from

Lemma A.1.3: $v \in \mathcal{E}'$, $\varphi \in S^m$, $d(x, \operatorname{supp} v) \gtrsim 1$
(beware of Shubin's notation)
$\Rightarrow |D^\alpha \varphi(D) v(x)| \lesssim C_{\alpha,N} |x|^{-N} \quad \forall N.$

Proof: Idea: $|\varphi(D)v(x)| = \Big|\int \underbrace{e^{i(x-y)\xi}}_{= |x-y|^{-2N}(-\Delta_\xi)^N e^{i(x-y)\xi}} \varphi(\xi) v(y)\, dy\, d\xi\Big|$

$$\overset{\text{int. by parts}}{=} \Big|\int e^{i(x-y)\xi} \underbrace{\big((-\Delta_\xi)^N \varphi(\xi)\big)}_{\lesssim C\langle\xi\rangle^{-n-1} \text{ for large } N} |x-y|^{-2N} v(y)\, dy\, d\xi\Big|$$

$$\lesssim C \int \langle\xi\rangle^{-n-1} d\xi \underbrace{\int dy \frac{v(y)}{|x-y|^{2N}}}_{\lesssim C\langle x\rangle^{-2N}} \qquad \square$$

See Shubin for details. The proof should remind you of the proof that the integral kernel is a smooth fct outside the diagonal, which decays rapidly as a fct of $x-y$ (see DiffFun 2).

Cor: For $A \in op(S^m)$ let $char(A) := \{(x,\xi) : a_m(x,\xi) = 0\}$

If $Au \in C^\infty$ $\Rightarrow$ $WF(u) \subseteq char(A)$.
(A prop. supp or $u \in \mathcal{E}'$)

Proof: $(x_0,\xi_0) \notin char(A)$ and $Au \in C^\infty \xRightarrow{\text{Prop. A.1.2}} (x_0,\xi_0) \notin WF(u)$. $\square$

Cor: $u \in \mathcal{E}'$ $(\mathcal{D}')$ $\Rightarrow$ $WF(u) = \bigcap_{\substack{A \in op(S^0) \\ Au \in C^\infty \\ (A\ prop.\ supp.)}} char(A)$.

In particular, $WF(u)$ is invariant under changes of coordinates.

Proof: „$\subseteq$" see above

„$\supseteq$" $(x_0,\xi_0) \notin WF(u) \xRightarrow{\text{Prop A.1.1}} \exists A \in op(S^0): a_m(x_0,\xi_0) = 1$, $Au \in C^\infty$

$\Rightarrow \exists A : (x_0,\xi_0) \notin char(A)$ $\square$

Prop A.1.3: $A \in op(S^m)$, $u \in \mathcal{D}'$, A prop. supp. or $u \in \mathcal{E}'$.

Then $a_m(x_0,\xi_0) \neq 0$, $(x_0,\xi_0) \notin WF(Au) \Longrightarrow (x_0,\xi_0) \notin WF(u)$

In other words, $WF(u) \subset char(A) \cup WF(Au)$

Proof: Prop A.1.1: $\exists$ prop. supp. $B \in op(S^0)$, $symbol(B) \sim 1$ in conical neighborhood of (x_0,ξ_0), $(\underbrace{BA}_{\text{elliptic near }(x_0,\xi_0)})u \in C^\infty$

$\xRightarrow{\text{Prop A.1.2}}$ $(x_0,\xi_0) \notin WF(u)$ $\square$

Prop A.1.4: (Pseudo locality)

$u \in \mathcal{D}'$, $A \in op(S^m_{1,0})$, A prop. supp. or $u \in \mathcal{E}'$.

Then $(x_0,\xi_0) \notin WF(u) \Longrightarrow (x_0,\xi_0) \notin WF(Au)$,

i.e. $WF(Au) \subset WF(u)$

Remark: By Lemma A.1.2: $sing\,supp(Au) \subset sing\,supp\, u$

Proof: $(x_0,\xi_0) \notin WF(u) \Rightarrow \exists$ prop. supp. $B \in op(S^0)$: $Bu \in C^\infty$, $symbol(B) \sim 1$ in conical neighborhood of (x_0,ξ_0).

$Q \in op(S^0)$ prop supp, $q_0(x_0,\xi_0) \neq 0$, symbol of Q rapidly decreasing outside small conical neighborhood of (x_0,ξ_0) s.t.

$BQ \sim Q$ if $QB \sim Q$.

$$\underset{\substack{\uparrow\\ \text{elliptic}\\ \text{near }(x_0,\xi_0)}}{Q} Au = \underbrace{QAB}_{\in C^\infty}u + \overbrace{\underset{\substack{\cap\\ \text{op}\,S^{-\infty}}}{R}\,u}^{\in C^\infty} \in C^\infty. \xRightarrow{\text{Prop A.1.2: } Q \text{ elliptic near } (x_0,\xi_0)} (x_0,\xi_0) \notin WF(Au) \qquad \square$$

Cor: $A \in \text{op}S^m \Rightarrow WF(Au) \underset{1.4}{\subset} WF(u) \underset{1.3}{\subset} WF(Au) \cup \text{char}(A)$

Cor: $A \in \text{op}S^m$ elliptic $\Rightarrow WF(u) = WF(Au)$

Proof: $\text{char}\,A = \emptyset$. $\square$

Applications:

① $u_1, u_2 \in \mathcal{D}'$. Q: $u_1 \cdot u_2 = ?$

In general: undefined. $\delta\cdot\delta$ does not make sense!

Assume $\boxed{WF(u_1) + WF(u_2) \subset X \times (\mathbb{R}^n \setminus \{0\})}$

(i.e. $(x,\xi) \in WF(u_1) \Rightarrow (x,-\xi) \notin WF(u_2)$)

Partition of 1: $1 = \sum_j \varphi_j^2$ $\qquad u_i^j = \varphi_j u_i$

Assume $\text{supp}\,\varphi_j$ small enough s.t. $|\mathcal{F}(\varphi_j u_i)| \le C_N (1+|\xi|)^{-N}$ for $\xi \in \mathbb{R}^n \setminus \Gamma_i$

Γ_i cones, $\Gamma_1 + \Gamma_2 \subset \mathbb{R}^n \setminus \{0\}$

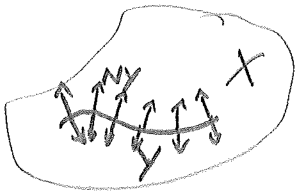
Define: $u_1^j u_2^j := \mathcal{F}^{-1}\left(\mathcal{F}(\varphi_j u_1) * \mathcal{F}(\varphi_j u_2)\right)$

$$= \mathcal{F}^{-1} \int \underbrace{\underbrace{\widehat{u_1\varphi_j}(\xi-\eta)}_{\substack{\text{rapidly decreasing}\\ \text{in } \mathbb{R}^n\setminus\Gamma_1}} \; \underbrace{\widehat{u_2\varphi_j}(\eta)}_{\substack{\text{— " —}\\ \text{in } \mathbb{R}^n\setminus\Gamma_2}}}_{\text{rapidly decreasing}} \, d\eta$$

$u_1 u_2 = \sum_j (\varphi_j u_1)(\varphi_j u_2)$

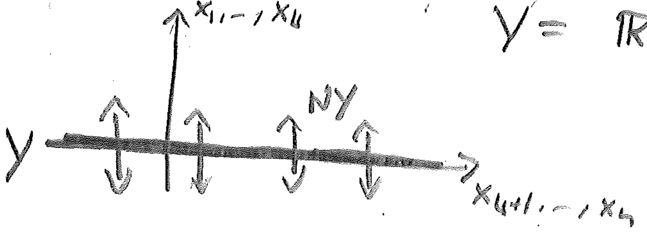
The product is uniquely defined by continuity.

② $u \in \mathcal{D}'(X)$, Y submanifold of X <u>Q</u>: $u|_Y = ?$



NY normal vectors to Y in X

Near a point $y \in Y$, we can write Y as the zero set of k fcts: $\{x_1 = \dots = x_k = 0\}$ so we only have to understand the case of $Y = \mathbb{R}^k \subset X = \mathbb{R}^n$:



Fourier inversion: $u \in \mathcal{S}(\mathbb{R}^n)$: $u|_Y = u(0, \dots, 0, x_{k+1}, \dots, x_n)$

$$= \mathcal{F}^{-1}_{(\xi_{k+1},\dots,\xi_n)\to(x_{k+1},\dots,x_n)} \mathcal{F}^{-1}_{(\xi_1,\dots,\xi_k)\to(x_1,\dots,x_k)} \mathcal{F}u \qquad x_1 = 0, \dots, x_k = 0$$

$$= \mathcal{F}^{-1}_{(\xi_{k+1},\dots,\xi_n)\to(x_{k+1},\dots,x_n)} \int d(\xi_1,\dots,\xi_k)\, \mathcal{F}u(\xi)$$

Now $u \in \mathcal{E}'(X)$, $\boxed{WF(u) \cap NY = \emptyset.}$

As $(\xi_1, \dots, \xi_k, 0, \dots, 0) \notin WF(u)$, $\mathcal{F}u$ rapidly decreasing in $\xi_1, \dots, \xi_k$

$\Rightarrow$ $\int d(\xi_1, \dots, \xi_k)\, \mathcal{F}u(\xi)$ exists, $\in \mathcal{S}'(Y)$

$\Rightarrow$ can take $\mathcal{F}^{-1}_{(\xi_{k+1},\dots,\xi_n)\to(x_{k+1},\dots,x_n)}$

$\Rightarrow$ $u|_Y$ for $u \in \mathcal{E}'(X)$. If $u \in \mathcal{D}'(X)$, define

$$u|_Y = \sum_j (\varphi_j u)|_Y, \quad \sum_j \varphi_j = 1.$$

Restriction is a special case of a pull-back $u \mapsto u \circ f$, where $f: Y \to X$ is the inclusion. Similarly, products are obtained by extending the restriction of smooth functions on $X \times X$ to the diagonal $\{(x,x) : x \in X\} =: \Delta$

$$C^\infty(X) \times C^\infty(X) \longrightarrow C^\infty(X \times X) \longrightarrow C^\infty(X)$$

$$(u, v) \longmapsto u(x)v(y) \longmapsto u(x)v(x) = u(x)v(y)|_\Delta .$$

In general we have the following theorem (Hörmander, The Analysis of Linear Partial Diff. Operators, Vol. I, Thm 8.2.4).

Thm: $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ open, $f: Y \to X$ C^∞, $N_f := \{ (f(y), \eta) \in Y \times (\mathbb{R}^n \setminus \{0\}) : {}^t f'(y)\eta = 0 \}$

$\Longrightarrow$ $\exists$ unique extension of $f^*: C^\infty(X) \to C^\infty(Y)$, $u \mapsto u \circ f$ by continuity to a map

$$f^*: \mathcal{D}'(X) \cap \{ u : N_f \cap WF(u) = \emptyset \} \longrightarrow \mathcal{D}'(Y)$$ and $WF(f^*u) \subset f^*(WF(u$