one has $\psi(\xi) + \sum_j a_j(x,\xi) = 1$ for $|x - x_0| < \epsilon$ and $\xi \in \mathbb{R}^n$. We then take $\Omega = \{x; |x - x_0| < \epsilon\}$; for any $\varphi \in C_0^\infty(\Omega)$, $\varphi = \varphi\psi + \sum_j \varphi a_j$ and

$$\varphi u = \varphi\psi(D)u + \varphi \sum_j a_j(x, D)u.$$

One has $\psi(D)u \in \mathcal{P}$ since $\psi \in S^{-\infty}$ (cf. Corollary 3.8), and $a_j(x, D)u \in C^\infty$ by assumption, so that $\varphi u \in C^\infty$. ∎

The next result links singularities of $a(x, D)u$ and singularities of u.

THEOREM 4.7 MICROLOCAL PROPERTY
Let $u \in \mathcal{S}'$ and $a \in S^\infty$; then

$$WF(a(x, D)u) \subset WFu \subset WF(a(x, D)u) \cup \mathit{Char}\ a.$$

PROOF Assume that $(x_0, \xi_0) \notin WF(a(x, D)u) \cup$ Char a. Then, there exist a conic neighborhood Γ of (x_0, ξ_0) and a $b \in S^\infty$ such that

$$c\#a(x, D)u \in C^\infty \quad \text{for all } c \in S^\infty_{\text{comp}}(\Gamma) \quad \text{and} \quad b\#a - 1 \in S^{-\infty}_{\text{loc}}(\Gamma).$$

Then for any $d \in S^\infty_{\text{comp}}(\Gamma)$, one has $d\#(1 - b\#a) \in S^{-\infty}$ and one can write $d\#b = c + r$ with $c \in S^\infty_{\text{comp}}(\Gamma)$ and $r \in S^{-\infty}$, from which one gets

$$d(x, D)u = d\#(1 - b\#a)(x, D)u + c\#a(x, D)u + r\#a(x, D)u.$$

The first and third terms are in $\mathcal{P}$ since the operators are in $\Psi^{-\infty}$, while the central term is also in C^∞ by assumption, so that $d(x, D)u \in C^\infty$.

Assume now that $(x_0, \xi_0) \notin WFu$, which means $b(x, D)u \in C^\infty$ for some conic neighborhood Γ of (x_0, ξ_0) and all $b \in S^\infty_{\text{comp}}(\Gamma)$. Then, for all $c \in S^\infty_{\text{comp}}(\Gamma)$, $c\#a = b+r$ with $b \in S^\infty_{\text{comp}}(\Gamma)$ and $r \in S^{-\infty}$, from which $c\#a(x, D)u = b(x, D)u + r(x, D)u$ where the two terms are smooth since $r \in S^{-\infty}$ and $b(x, D)u \in C^\infty$ by assumption. ∎

Let $a(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ be a linear differential operator with *real-valued* coefficients $a_\alpha \in C^\infty$ in the principal part (i.e., for $|\alpha| = m$), and $u \in \mathcal{S}'$ a solution of $a(x, D)u = f$ where $f \in C^\infty$. Thanks to Theorem 4.7 one has $WFu \subset$ Char $a = \{(x, \xi); p(x, \xi) = 0\}$ where $p(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is the principal symbol of a. Now, we are going to prove a more precise result, namely that WFu is actually a union of curves, called "bicharacteristic curves," contained in Char a: this property is known as the "propagation of singularities."

Let $p \in C^\infty(T^*\mathbb{R}^n)$ be a real-valued function; its Hamiltonian vector field $H_p = \langle \partial_\xi p, \partial_x \rangle - \langle \partial_x p, \partial_\xi \rangle$ is clearly tangent to the level surfaces of p since one has $H_p p = \langle \partial_\xi p, \partial_x p \rangle - \langle \partial_x p, \partial_\xi p \rangle = 0$. Its integral curves, that is, the solutions $(x(t), \xi(t))$ of the equations

$$dx/dt = \partial_\xi p(x,\xi) \qquad d\xi/dt = -\partial_x p(x,\xi)$$

(which exist locally thanks to the Cauchy–Lipschitz theorem) are therefore contained in the level surfaces of p. In particular, those curves that start at a point $(x(0), \xi(0))$ where p vanishes satisfy $p(x(t), \xi(t)) = 0$, and they are then called *bicharacteristic curves of p*. If p and $q \in C^\infty(T^*\mathbb{R}^n)$ are real valued and if $p(x_0, \xi_0) = 0$ while $q(x_0, \xi_0) \neq 0$, then one has $H_{pq} = qH_p$ on $p = 0$ near (x_0, ξ_0) so that bicharacteristic curves of p or of pq are geometrically the same near (x_0, ξ_0), although parameterized in a different way. One can then state the propagation result.

THEOREM 4.8 PROPAGATION THEOREM
Let $a(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be a linear differential operator with a real-valued principal symbol $p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$. If $u \in S'$ satisfies an equation $a(x, D)u = f$ with $f \in C^\infty$, then its wave front set is a union of bicharacteristic curves of p.

Before giving the (rather long) proof of this theorem, we will prove a technical lemma that forms the basis of the study of the Cauchy problem for hyperbolic operators with variable coefficients. In return, this latter problem will then be the typical situation where the result of Theorem 4.8 can be used, as we will see in Section 4.3. If I is a compact interval of $\mathbb{R}$, we will use the notation $C^0(I; H^k)$ for the space of functions $w(t)$, which are continuous functions of t valued in the Sobolev space H^k. Similarly, we will use $C^1(I; H^k)$ and $C^0(I; S)$ in the same way.

LEMMA 4.9
Let $b \in S^1$ be a symbol satisfying $b - b^* \in S^0$, $s > 0$, and $I = [-s, s]$; then for any $k \in \mathbb{Z}$ one has

$$\sup_{t\in I} \|e^{\mu t} w(t)\|_k \leq \|e^{\mu s} w(s)\|_k + 2\int_I \|e^{\mu t}(\partial_t - ib(x, D))w(t)\|_k \, dt,$$

and

$$\sup_{t\in I} \|e^{-\mu t} w(t)\|_k \leq \|e^{\mu s} w(-s)\|_k + 2\int_I \|e^{-\mu t}(\partial_t - ib^*(x, D))w(t)\|_k \, dt$$

for a constant μ depending on k and for all $w \in C^0(I; H^{k+1}) \cap C^1(I; H^k)$. Moreover, for any $g \in C^0(I; S)$ and $\chi \in S$, the Cauchy problem

$$\begin{cases} (\partial_t - ib(x, D))w(t) = g(t) \\ w(s) = \chi \end{cases}$$

has only one solution $w \in \cup_k C^0(I; H^k)$, and this solution actually satisfies $w \in \cap_k C^0(I; H^k)$.

PROOF We will give the proof in three steps.

(i) *Energy estimates.* Since $b - b^* \in S^0$, one has for any $w \in C^0(I; H^1)$

$$|2\mathrm{Re}\,(e^{\mu t} ib(x,D)w(t), e^{\mu t}w(t))| = |(i(b-b^*)(x,D)e^{\mu t}w(t), e^{\mu t}w(t))| \leq C\|e^{\mu t}w(t)\|_0^2.$$

Thus one can write for $w \in C^0(I; H^1) \cap C^1(I; H^0)$

$$\begin{aligned}\frac{d}{dt}\|e^{\mu t}w(t)\|_0^2 &= 2\mu\|e^{\mu t}w(t)\|_0^2 + 2\mathrm{Re}\,(e^{\mu t}\partial_t w(t), e^{\mu t}w(t))\\ &\geq (2\mu - C)\|e^{\mu t}w(t)\|_0^2 + 2\mathrm{Re}\,(e^{\mu t}(\partial_t - ib(x,D))w(t), e^{\mu t}w(t))\\ &\geq -2\|e^{\mu t}(\partial_t - ib(x,D))w(t)\|_0\|e^{\mu t}w(t)\|_0\end{aligned}$$

if one has chosen $\mu \geq C/2$. Using the notation $W(t) = \|e^{\mu t}w(t)\|_0$, $M = \sup_{t\in I} W(t)$, and $\mathrm{Int} = \int_I \|e^{\mu t}(\partial_t - ib(x,D))w(t)\|_0\,dt$, we can integrate this estimate over the interval $[t,s]$ then take the supremum for $t \in I$, and this gives $M^2 \leq W(s)^2 + 2M\mathrm{Int}$, so that $(M - \mathrm{Int})^2 \leq W(s)^2 + \mathrm{Int}^2$ then $M - \mathrm{Int} \leq W(s) + \mathrm{Int}$, which is our first estimate with $k = 0$.

For $k \neq 0$ one has

$$(\partial_t - i(\lambda^k \# b \# \lambda^{-k})(x,D))\lambda^k(D)w(t) = \lambda^k(D)(\partial_t - ib(x,D))w(t),$$

and since $b_k = \lambda^k \# b \# \lambda^{-k} \in S^1$ still satisfies $b_k - b_k^* \in S^0$, one can use the previous result (now μ depends on k), which gives

$$\sup_{t\in I}\|e^{\mu t}\lambda^k(D)w(t)\|_0 \leq \|e^{\mu s}\lambda^k(D)w(s)\|_0 + 2\int_I \|e^{\mu t}\lambda^k(D)(\partial_t - ib(x.D))w(t)\|_0\,dt,$$

and this is exactly our first energy estimate.

Finally, the second estimate in terms of $w(-s)$ and b^* can be obtained in the same way by changing t in $-t$ since b^* also satisfies $b^* - (b^*)^* \in S^0$.

(ii) *Uniqueness.* If the Cauchy problem had two solutions in $\cup_k C^0(I; H^k)$ for the same data g and χ, then their difference $w(t)$ would satisfy $w \in C^0(I; H^{k+1})$ for some $k \in \mathbb{Z}$, $\partial_t w(t) = ib(x,D)w(t)$, and $w(s) = 0$. The equation for $\partial_t w(t)$ shows that $w \in C^1(I; H^k)$, then the energy estimate implies that $w(t) = 0$ for $t \in I$, and this is the uniqueness property.

With the second energy estimate, one would prove in the same way that the Cauchy problem

$$\begin{cases}(\partial_t - ib^*(x,D))\psi(t) = \varphi(t)\\ \psi(-s) = 0\end{cases}$$

has at most one solution $\psi \in \cup_k C^0(I; H^k)$.

(iii) *Existence and smoothness of the solution.* Actually, we will show that for any $g \in C^0(I;\mathcal{S})$, $\chi \in \mathcal{S}$, and $k \in \mathbb{Z}$, the Cauchy problem with data g and χ has a solution $w_k \in C^0(I;H^{k-1})$. However, thanks to the uniqueness property we got in step (ii), all these solutions w_k are equal to a unique solution $w \in \cap_k C^0(I;H^k)$.

For any φ taken in the space

$$\mathbb{E} = \{(\partial_t - ib^*(x,D))\psi(t); \psi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n) \text{ with supp } \psi \subset \{(t,x); t > -s\}\},$$

the Cauchy problem

$$\begin{cases} (\partial_t - ib^*(x,D))\psi(t) = \varphi(t) \\ \psi(-s) = 0 \end{cases}$$

has a solution $\psi \in C^0(I;\mathcal{S})$, namely the restriction to I of the ψ given in the definition of $\mathbb{E}$. This solution is unique according to step (ii) and it satisfies the energy estimate $\sup_{t\in I}\|\psi(t)\|_{-k} \le C_k \int_I \|\varphi(t)\|_{-k}\,dt$ for any $k \in \mathbb{Z}$. Then, for $g \in C^0(I;\mathcal{S})$ and $\chi \in \mathcal{S}$, one defines a semilinear form W on $\mathbb{E}$ by

$$W(\varphi) = (\chi, \psi(s)) - \int_I (g(t), \psi(t))\,dt$$

where ψ is the unique solution of the previous Cauchy problem. One has

$$|W(\varphi)| \le \left(\|\chi\|_k + \int_I \|g(t)\|_k\,dt\right) \sup_{t\in I}\|\psi(t)\|_{-k} \le C'_k \int_I \|\varphi(t)\|_{-k}\,dt,$$

so that W is continuous for the norm of the space $L^1(I;H^{-k})$, and thanks to the Hahn–Banach theorem, there exists an element $w \in L^\infty(I;H^k) \simeq (L^1(I;H^{-k}))'$ such that

$$(\chi, \psi(s)) - \int_I (g(t), \psi(t))\,dt = \int_I (w(t), (\partial_t - ib^*(x,D))\psi(t))\,dt$$

for all $\psi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$ with supp $\psi \subset \{(t,x); t > -s\}$.

When we restrict ourselves to functions $\psi \in C_0^\infty$ with supp $\psi \subset \Omega = \{(t,x); |t| < s\}$, the term $(\chi, \psi(s))$ vanishes and the previous equation then means that $(\partial_t - ib(x,D))w(t) = g(t)$ in Ω. It follows that $\partial_t w \in L^\infty(I;H^{k-1})$ whence $w \in C^0(I;H^{k-1})$, and similarly $w \in C^1(I;H^{k-2})$. Moreover, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ one can construct a $\psi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$ with supp $\psi \subset \{(t,x); t > -s\}$ and $\psi(s) = \varphi$, so that the integration of $\int_I (w(t), \partial_t\psi(t))\,dt$ by parts gives $(w(s),\varphi) = (\chi,\varphi)$, and this proves that the function $w \in C^0(I;H^{k-1})$ is a solution to the Cauchy problem with data g and χ. ∎

REMARK Of course, we could have solved the Cauchy problem with data on $t = -s$ as well since all the assumptions are preserved when reversing the time. To use this result in the study of the Cauchy problem for a hyperbolic operator with variable coefficients, we would have to consider more generally symbols b depending also on t, but this would not affect the proof very much (cf. Hörmander [8, Chap. 23], where we took the proof of Lemma 4.9). ∎

PROOF OF THEOREM 4.8 Since WFu is a closed set and a bicharacteristic curve γ is a connected set, the only fact to prove is that $WFu \cap \gamma$ is an open subset of γ. Thus, given a $(x_0, \xi_0) \in WFu \subset \text{Char } a$, we want to prove that the bicharacteristic curve $(x(t), \xi(t))$ starting at this point is locally contained in WFu. We will proceed by contradiction, assuming that this is not true.

Since the action of an operator elliptic at (x_0, ξ_0) does not modify WFu near this point (cf. Theorem 4.7), we will deal with the distribution $v = \lambda^{m-1}(D)(\psi u)$ rather than with u itself where the function $\psi \in C_0^\infty$ satisfies $\psi = 1$ near x_0. The advantage is twofold: on one hand, we have $\psi u \in \mathcal{E}'$ from which we get $r(x, D)v \in \mathcal{S}$ for all $r \in S^{-\infty}$ (cf. Corollary 3.8). On the other hand, if $\varphi \in C_0^\infty$ is chosen such that $\varphi = 1$ near x_0 and $\psi = 1$ near $\operatorname{supp} \varphi$, the distribution v will satisfy an equation $b(x, D)v = \varphi f \in C_0^\infty$ where $b = \varphi a \lambda^{1-m}$ is a first-order polyhomogeneous symbol (the three factors are polyhomogeneous) whose principal symbol $q(x, \xi) = \varphi(x)p(x, \xi)|\xi|^{1-m}$ is real valued and has the same bicharacteristic curves as p.

Our contradiction will then be that $(x_0, \xi_0) \notin WFv = WFu$, and this will follow from the construction of a symbol $c_0 \in S^0$ *elliptic at* (x_0, ξ_0) and such that $c_0(x, D)v \in C^\infty$. To prove this latter property, we will construct a symbol $c(t, x, \xi)$ satisfying $c(0, x, \xi) = c_0(x, \xi)$ and such that $w(t) = c(t, x, D)v$ is a solution of a Cauchy problem as in Lemma 4.9. Since this $w \in C^0(I; H^{-N})$ for some $N \in \mathbb{Z}_+$, it will then follow from Lemma 4.9 that $w \in \cap_k C^0(I; H^k)$, so that $c_0(x, D)v = w(0) \in C^\infty$.

The symbol $c(t, x, \xi)$, a C^1 function of t valued in S^0, is going to be constructed so that

$$r(t) = \partial_t c(t) - i(b \# c(t) - c(t) \# b)$$

is a continuous function of t valued in $S^{-\infty}$. To achieve this, we just have to choose a c_0 homogeneous in ξ of degree zero, and to look for a polyhomogeneous symbol $c(t, x, \xi) \sim \sum_j c_j(t, x, \xi)$ where the c_j are homogeneous in ξ of degree $-j$. Following this point of view and ordering the terms in the asymptotic expansions of $b \# c(t)$ and $c(t) \# b$ according to their degrees of homogeneity, we then have to solve the sequence of problems

$$\begin{cases} \partial_t c_0 - H_q c_0 = 0 \\ c_0(0, x, \xi) = c_0(x, \xi), \end{cases} \qquad \text{and for } j > 0 \begin{cases} \partial_t c_j - H_q c_j = r_j \\ c_j(0, x, \xi) = 0 \end{cases}$$

where the vector field $\partial_t - H_q$ is real. These problems can be solved in a same neighborhood of (x_0, ξ_0) thanks to the Cauchy–Lipschitz theorem, and their solutions $c_j(t)$ are there homogeneous in ξ of degree $-j$ and have their support contained in the support of $c_0(t)$. If we have chosen $c_0 \in S^0_{\text{comp}}(\Gamma)$ for some small conic neighborhood Γ of (x_0, ξ_0), it thus follows that for small t, the $c_j(t)$ can be extended by homogeneity as symbols $c_j(t) \in S^{-j}_{\text{comp}}(\Gamma)$, and the symbol $c(t, x, \xi)$ is then constructed as in Lemma 2.2. Since we assume that the bicharacteristic curve is not locally contained in WFu, there is a point $(x(s), \xi(s)) \notin WFu$ for an s in the domain of definition of c, and one has

$$\text{supp } c(s) \subset \text{ supp } c_0(s) \subset e^{sH_q} \text{ supp } c_0$$

where e^{sH_q} supp c_0 denotes the image of supp c_0 under the flow of the Hamiltonian vector field H_q. Therefore, if the (conic) support of c_0 has been chosen sufficiently small around (x_0, ξ_0), the support of $c(s)$ will be small around $(x(s), \xi(s))$ and we will have $w(s) = c(s, x, D)v = \chi \in C^\infty$ (and actually $\chi \in C_0^\infty$ since the support of $c(s)$ is compact in x).

Thanks to Lemma 1.16, the distribution ψu is in some Sobolev space so that $v = \lambda^{m-1}(D)(\psi u) \in H^{-N}$ for some $N \in \mathbb{Z}_+$ and then $w(t) = c(t, x, D)v \in C^0(I; H^{-N})$ since $c \in C^0(I; S^0)$. Moreover, this distribution $w(t)$ satisfies $\partial_t w(t) - ib(x, D)w(t) = g(t)$ where

$$g(t) = r(t, x, D)v - ic(t, x, D)b(x, D)v$$

by construction of $c(t)$, and these two terms are in $C^0(I; \mathcal{S})$ because $r \in C^0(I; S^{-\infty})$ and $v = \lambda^{m-1}(D)(\psi u)$ with $\psi u \in \mathcal{E}'$ (cf. Corollary 3.8), and $b(x, D)v = \varphi f \in C_0^\infty$. It follows that w is a $C^0(I; H^{-N})$ solution of a Cauchy problem as in Lemma 4.9 (indeed, $b \in S^1$ satisfies $b - b^* \in S^0$ since it is polyhomogeneous with a real principal symbol), therefore $w \in \cap_k C^0(I; H^k)$, then $c_0(x, D)v = w(0) \in H^\infty \subset C^\infty$ as claimed. ∎

4.3 The Cauchy problem for the wave equation

This section actually has little to do with pseudodifferential operators. Following an idea of Treves [12], the Cauchy problem for the wave equation is treated through very simple computations, which allow us to prove the main results. We have added this section because the last theorem provides a pleasant illustration of the general results given in the previous section.

Consider in the space $\mathbb{R}^{n+1} = \{(t, x); t \in \mathbb{R}, x \in \mathbb{R}^n\}$ the wave operator, i.e., the operator $u \mapsto \partial_t^2 u - \Delta u$ where $\Delta = \sum_{j=1}^n \partial_j^2$ with $\partial_j = \partial/\partial x_j$ is the Laplacian operator in $\mathbb{R}^n$. The present treatment will be completely dissymmetric in x and t: we will consider solutions u such that for each fixed t, $u(t) \in \mathcal{S}'$ (space of temperate distributions in x only).

The spaces $C^k(\mathbb{R}; H^s)$ and $C^k(\mathbb{R}; \mathcal{S})$ are the standard spaces of C^k functions of t valued in the Hilbert space H^s (Sobolev space in x only) or in the Fréchet space $\mathcal{S}$ (Schwartz space in x only). We will write $u \in C^0(\mathbb{R}; \mathcal{S}')$ if for each fixed $\varphi \in \mathcal{S}$ the expression $(u(t), \varphi)$ is a continuous function of t, and actually,