Since $\hat{\varphi}(\xi) \in \mathscr{S}_\xi$, from (5.12) for $\eta = 0$ and (5.10) we obtain

$$(\mathscr{F}[T], \hat{\varphi})_\xi = \int_{\mathbf{R}^n_\xi} (T, e^{ix\cdot\xi})_x \overline{\hat{\varphi}(\xi)} đ\xi$$

which means that $\mathscr{F}[T] = (T, e^{ix\cdot\xi})_x$. Thus we get (5.8) and (5.9) from (5.12). ■

Example 1° $\mathscr{F}[\delta] = 1$ in $\mathbf{R}^n$. This is clear from

$$(\mathscr{F}[\delta], \hat{\varphi})_\xi = (\delta, \varphi)_x = \overline{\varphi(0)} = \int 1 \cdot \overline{\hat{\varphi}(\xi)} đ\xi = (1, \hat{\varphi})_\xi.$$

Example 2° Using the Heaviside function $Y(\xi)$ defined by $Y(\xi) = 1$ $(\xi > 0)$, $= 0$ $(\xi \leqq 0)$, we obtain

$$\mathscr{F}\left[\frac{1}{x \pm i\tau}\right](\xi) = \mp 2\pi i e^{-\tau|\xi|} Y(\pm \xi) \qquad (\tau > 0), \tag{5.13i}$$

$$\mathscr{F}\left[\frac{1}{x \pm i0}\right](\xi) = \mp 2\pi i Y(\pm \xi). \tag{5.13ii}$$

In fact, since $\dfrac{1}{x \pm i\tau} \in L_2(\mathbf{R}^1_x)$, we get from (2) of Theorem 4.5

$$\mathscr{F}\left[\frac{1}{x \pm i\tau}\right](\xi) = \lim_{R\to\infty} \int_{-R}^{R} e^{-ix\cdot\xi} \frac{1}{x \pm i\tau} dx \quad \text{in} \quad L_2(\mathbf{R}^1_\xi).$$

Corresponding to $\xi \gtrless 0$ we take the path of integration as in Figure 1, and calculate the residues. Then we get (5.13i) and letting $\tau \to 0$, we get (ii).

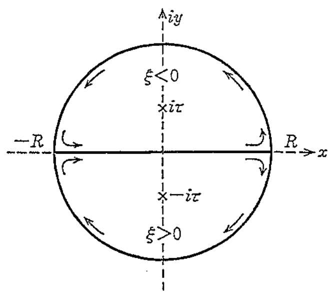

Figure 1

H. Kumano-go, Pseudodifferential Operators, MIT Press (1974)

Example 3°

$$\mathscr{F}\left[\text{p.v.}\frac{1}{x}\right] = -\pi i \operatorname{sign} \xi \quad (\operatorname{sign} \xi = 1\ (\xi > 0),\ = -1\ (\xi < 0)).$$

In fact, from (2.25) and (5.13ii),

$$\mathscr{F}\left[\text{p.v.}\frac{1}{x}\right] = \mathscr{F}\left[\frac{1}{2}\left(\frac{1}{x + i0} + \frac{1}{x - i0}\right)\right] = -\pi i \operatorname{sign} \xi.$$

Hence, using (5.7), for the Hilbert transform $H\varphi = \varphi * \left(\text{p.v.}\dfrac{1}{x}\right)$ of $\varphi \in \mathscr{S}_x$ we obtain

$$\mathscr{F}\left[\varphi * \left(\text{p.v.}\frac{1}{x}\right)\right] = \mathscr{F}\left[\lim_{\varepsilon\to 0} \int_{|y|\geq\varepsilon} \frac{\varphi(x-y)}{y} dy\right] = -(\pi i \operatorname{sign} \xi)\hat{\varphi}(\xi). \tag{5.14}$$

§6 Oscillatory Integrals

The term *oscillatory integral* has been introduced by Lax [1] to designate integrals of the form

$$Af(x) = \int e^{iS(x,\xi)} a(x, \xi) \hat{f}(\xi) đ\xi,$$

which are used for the construction of asymptotic solutions of hyperbolic equations (see §§1 and 4 of Chapter 10). The function $S(x, \xi)$ is called the *phase function*, and $a(x, \xi)$ is called the *amplitude function*. Exploiting this idea, Hörmander in [6] and [9] used integrals of the form

$$Af(x) = \iint e^{i\varphi(x,\eta,y)} a(x, \eta, y) f(y) dy đ\eta, \tag{6.1}$$

which he called *Fourier integral operators*, to study the regularity of solutions of partial differential equations. We shall present this theory in Chapter 10.

In the present section we take $\varphi(x, \eta, y)$ in (6.1) to be $\varphi = -y\cdot\eta = -(y_1\eta_1 + \cdots + y_n\eta_n)$, and we define oscillatory integrals of the form

$$\text{O}_s - \iint e^{-iy\cdot\eta} a(\eta, y) dy đ\eta$$

for amplitude functions $a(\eta, y)$ of polynomial growth. The key point of the theory lies in the fact that the integration is always done over the pair $dy\bar{d}\eta$. The fundamental theorems corresponding to those of the usual integration theory can be derived by means of integrations by parts. These theorems will be used mainly in Chapter 2.

First we define the class of amplitude functions.

DEFINITION 6.1 We say that a C^∞-function $a(\eta, y)$ in $\mathbf{R}^{2n}_{\eta,y} = \mathbf{R}^n_\eta \times \mathbf{R}^n_y$ belongs to the class $\mathscr{A}^m_{\delta,\tau}$ $(-\infty < m < \infty, 0 \leqq \delta < 1, 0 \leqq \tau)$ if for any multi-indices α, β there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial^\alpha_\eta \partial^\beta_y a(\eta, y)| \leqq C_{\alpha,\beta} \langle\eta\rangle^{m+\delta|\beta|} \langle y\rangle^\tau. \tag{6.2}$$

We set $\mathscr{A} = \bigcup_{0\leqq\delta<1} \bigcup_{-\infty<m<\infty} \bigcup_{0\leqq\tau} \mathscr{A}^m_{\delta,\tau}$.

Then, the following facts are clear.

1° If for $a(\eta, y) \in \mathscr{A}^m_{\delta,\tau}$ we define the semi-norms $|a|_l$, $l = 0, 1, \ldots$, by

$$|a|_l = \max_{|\alpha+\beta|\leqq l} \sup_{(\eta,y)} \{|\partial^\alpha_\eta \partial^\beta_y a(\eta, y)| \langle\eta\rangle^{-(m+\delta|\beta|)} \langle y\rangle^{-\tau}\},$$

then $\mathscr{A}^m_{\delta,\tau}$ is a Fréchet space, and for any $a(\eta, y) \in \mathscr{A}^m_{\delta,\tau}$ we have

$$|\partial^\alpha_\eta \partial^\beta_y a(\eta, y)| \leqq |a|_l \langle\eta\rangle^{m+\delta|\beta|} \langle y\rangle^\tau \qquad (l = |\alpha + \beta|). \tag{6.2'}$$

2° For $a, a_1, a_2 \in \mathscr{A}$ we have

$$\partial^\alpha_\eta \partial^\beta_y a,\ a_1 + a_2,\ a_1 a_2 \in \mathscr{A}.$$

DEFINITION We say that a subset B of $\mathscr{A}$ is a *bounded set* of $\mathscr{A}$, if there exists $\mathscr{A}^m_{\delta,\tau}$ such that

$$B \subset \mathscr{A}^m_{\delta,\tau} \quad \text{and} \quad \sup_{a\in B} \{|a|_l\} < \infty \tag{6.3}$$

holds for any $l = 0, 1, \ldots$.

Remark In general a subset B of a linear topological space V is said to be a bounded set of V, if for any neighborhood U of the origin there exists a $\lambda > 0$ such that $B \subset \lambda U$. Hence, a subset B of $\mathscr{A}$ is bounded, if and only if B lies in a Fréchet space $\mathscr{A}^m_{\delta,\tau}$ and it is a bounded set of $\mathscr{A}^m_{\delta,\tau}$. In this work we shall need only the notion of bounded set in $\mathscr{A}$.

DEFINITION 6.2 For an element $a(\eta, y)$ of $\mathscr{A}$ we define the oscillatory integral $\mathrm{O_s}[e^{-iy\cdot\eta}a]$ by

$$\begin{aligned} \mathrm{O_s}[e^{-iy\cdot\eta}a] &\equiv \mathrm{O_s} - \iint e^{-iy\cdot\eta} a(\eta, y) dy\bar{d}\eta \\ &= \lim_{\varepsilon\to 0} \iint e^{-iy\cdot\eta} \chi(\varepsilon\eta, \varepsilon y) a(\eta, y) dy\bar{d}\eta, \end{aligned} \tag{6.4}$$

where $\chi(\eta, y) \in \mathscr{S}$ in $\mathbf{R}^{2n}_{\eta,y}$ and $\chi(0, 0) = 1$.

In Theorem 6.4 we show that $\mathrm{O_s}[e^{-iy\cdot\eta}a]$ is well-defined, i.e., the limiting value in (6.4) is independent of the choice of $\chi \in \mathscr{S}$ satisfying $\chi(0, 0) = 1$. This fact will be established by means of the following Lemma 6.3, which will be used throughout this book. When $a(y, \eta)$ belongs to $L_1(\mathbf{R}^{2n}_{\eta,y})$, from the Lebesgue dominated convergence theorem it is clear that $\mathrm{O_s}[e^{-iy\cdot\eta}a]$ coincides with the usual value $\iint e^{-iy\cdot\eta} a(\eta, y) dy\bar{d}\eta$.

LEMMA 6.3 Let $\chi(x) \in \mathscr{S}$ in $\mathbf{R}^n_x$ with $\chi(0) = 1$. Then

$$\begin{aligned} &\chi(\varepsilon x) \to 1 \text{ in } \mathbf{R}^n \text{ uniformly on any compact set,} \\ &\partial^\alpha_x \chi(\varepsilon x) \to 0 \text{ in } \mathbf{R}^n \text{ uniformly } (\alpha \neq 0) \end{aligned} \tag{6.5}$$

as $\varepsilon \to 0$, and for any α there exists a constant C_α independent of $0 < \varepsilon < 1$ such that

$$|\partial^\alpha_x \chi(\varepsilon x)| \leqq C_\alpha \varepsilon^\sigma \langle x\rangle^{-(|\alpha|-\sigma)} \qquad (0 \leqq \sigma \leqq |\alpha|). \tag{6.6}$$

Proof Since $\partial^\alpha_x \chi(\varepsilon x) = \varepsilon^{|\alpha|} \partial^\alpha_y \chi(y)_{|y=\varepsilon x}$, (6.5) is clear. If we write

$$|\partial^\alpha_x \chi(\varepsilon x)| = \varepsilon^\sigma \{\varepsilon^{(|\alpha|-\sigma)} |\partial^\alpha_y \chi(y)_{|y=\varepsilon x}|\},$$

then (6.6) is clear when $|x| \leqq 1$. When $|x| > 1$, using

$$\begin{aligned} \varepsilon^{(|\alpha|-\sigma)} |\partial^\alpha_y \chi(y)_{|y=\varepsilon x}| &= (|y|^{(|\alpha|-\sigma)} |\partial^\alpha_y \chi(y)|)_{|y=\varepsilon x} |x|^{-(|\alpha|-\sigma)} \\ &\leqq C_\alpha \langle x\rangle^{-(|\alpha|-\sigma)} \qquad (0 \leqq \sigma \leqq |\alpha|) \end{aligned}$$

we also get (6.6). ∎

THEOREM 6.4 For $a \in \mathscr{A}$ the value of the oscillatory integral $\mathrm{O_s}[e^{-iy\cdot\eta}a]$ is independent of the choice of $\chi \in \mathscr{S}$ satisfying $\chi(0, 0) = 1$. When $a \in \mathscr{A}^m_{\delta,\tau}$, taking positive integers l and l' such that

$$-2l(1 - \delta) + m < -n, \quad -2l' + \tau < -n, \tag{6.7}$$

we have

$$|\langle y\rangle^{-2l'} \langle D_\eta\rangle^{2l'} \{\langle\eta\rangle^{-2l} \langle D_y\rangle^{2l} a(\eta, y)\}| \in L_1(\mathbf{R}^{2n}_{\eta,y}). \tag{6.8}$$

and we can write

$$\mathrm{O_s}[e^{-iy\cdot\eta}a] = \iint e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}a(\eta, y)\}dy\bar{d}\eta. \tag{6.9}$$

Furthermore there exists a constant C independent of $a\in\mathscr{A}^m_{\delta,\tau}$ such that

$$|\mathrm{O_s}[e^{-iy\cdot\eta}a]| \leqq C|a|_{l_0} \quad \text{for} \quad a\in\mathscr{A}^m_{\delta,\tau} \qquad (l_0 = 2(l+l')), \tag{6.10}$$

where

$$\langle D_\eta\rangle^{2l'} = (1-\Delta_\eta)^{l'}\left(=\left(1+\sum_{j=1}^{n} D_{\eta_j}^2\right)^{l'}\right), \quad \langle D_y\rangle^{2l} = (1-\Delta_y)^l.$$

Remark 1° It is clear that the oscillatory integral is a linear operation:

$$a_j\in\mathscr{A},\ \lambda_j\in\mathbf{C}\ (j=1,2) \Rightarrow \mathrm{O_s}[e^{-iy\cdot\eta}(\lambda_1 a_1 + \lambda_2 a_2)] = \lambda_1\mathrm{O_s}[e^{-iy\cdot\eta}a_1] + \lambda_2\mathrm{O_s}[e^{-iy\cdot\eta}a_2].$$

Remark 2° The estimate (6.10) shows that the mapping $\mathscr{A}^m_{\delta,\tau}\ni a\mapsto \mathrm{O_s}[e^{-iy\cdot\eta}a]\in\mathbf{C}$ is continuous.

Proof Using $D_y^\alpha e^{-iy\cdot\eta} = (-\eta)^\alpha e^{-iy\cdot\eta}$ and $D_\eta^\beta e^{-iy\cdot\eta} = (-y)^\beta e^{-iy\cdot\eta}$ we have

$$\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}e^{-iy\cdot\eta} = e^{-iy\cdot\eta}, \quad \langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}e^{-iy\cdot\eta} = e^{-iy\cdot\eta}. \tag{6.11}$$

Since $\chi(\varepsilon\eta, \varepsilon y)\in\mathscr{S}$ for any fixed $0<\varepsilon<1$, we can integrate by parts and obtain

$$\begin{aligned} I_\varepsilon &\equiv \iint e^{-iy\cdot\eta}\chi(\varepsilon\eta, \varepsilon y)a(\eta, y)dy\bar{d}\eta \\ &= \iint e^{-iy\cdot\eta}\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}(\chi(\varepsilon\eta, \varepsilon y)a(\eta, y))dy\bar{d}\eta \\ &= \iint e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}(\chi(\varepsilon\eta, \varepsilon y)a(\eta, y))\}dy\bar{d}\eta. \end{aligned}$$

On the other hand, by Lemma 6.3 $\{\chi(\varepsilon\eta, \varepsilon y)\}_{0<\varepsilon<1}$ is bounded in $\mathscr{A}^0_{0,0}$. So for any α, β there exists a constant $C_{\alpha,\beta}$ independent of both $0<\varepsilon<1$ and $a\in\mathscr{A}^m_{\delta,\tau}$ such that

$$|\partial_\eta^\alpha\partial_y^\beta(\chi(\varepsilon\eta, \varepsilon y)a(\eta, y))| \leqq C_{\alpha,\beta}|a|_{|\alpha+\beta|}\langle\eta\rangle^{m+\delta|\beta|}\langle y\rangle^\tau. \tag{6.12}$$

If for a real s we use $\partial_{\eta_j}\langle\eta\rangle^s = s\eta_j\langle\eta\rangle^{s-2}$, $j=1, \ldots, n$, then by induction there exists a constant $C_{s,\alpha}$ such that

$$|\partial_\eta^\alpha\langle\eta\rangle^s| \leqq C_{s,\alpha}\langle\eta\rangle^{s-|\alpha|}. \tag{6.13}$$

From (6.12) and (6.13) there exists a constant $C_{l,\alpha}$ such that

$$|\partial_\eta^\alpha\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}(\chi(\varepsilon\eta, \varepsilon y)a(\eta, y))\}| \leqq C_{l,\alpha}|a|_{2l+|\alpha|}\langle\eta\rangle^{m-2l(1-\delta)}\langle y\rangle^\tau.$$

Consequently there exists a constant $C_{l,l'}$ independent of $0<\varepsilon<1$ and $a\in\mathscr{A}^m_{\delta,\tau}$ such that

$$\begin{aligned} &|\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}(\chi(\varepsilon\eta, \varepsilon y)a(\eta, y))\}| \\ &\leqq C_{l,l'}|a|_{l_0}\langle\eta\rangle^{m-2l(1-\delta)}\langle y\rangle^{-2l'+\tau} \qquad (l_0 = 2(l+l')). \end{aligned} \tag{6.14}$$

Then by Condition (6.7) the right side of (6.14) belongs to $L_1(\mathbf{R}^{2n}_{\eta,y})$. Hence, by (6.5) and by the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} \mathrm{O_s}[e^{-iy\cdot\eta}a] &= \lim_{\varepsilon\to 0} I_\varepsilon \\ &= \iint e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}a(\eta, y)\}dy\bar{d}\eta, \end{aligned}$$

which means that (6.9) holds. Since the representation (6.9) does not involve χ, the value of $\mathrm{O_s}[e^{-iy\cdot\eta}a]$ is independent of the choice of χ satisfying $\chi(0,0)=1$. Finally, letting $\varepsilon\to 0$, (6.10) follows from (6.14) and (6.9). ∎

Now we prove an elementary inequality for C^2-functions defined on the interval $[0, 1]$.

LEMMA 6.5 Let $f(t)$ be a C^2-function defined on $I=[0,1]$. Then there exists a constant $M>0$ independent of f such that

$$(\max_I|f'(t)|)^2 \leqq M(\max_I|f(t)|)\{\max_I|f(t)| + \max_I|f''(t)|\}. \tag{6.15}$$

Proof We may assume that $f(t)$ is real-valued and $\not\equiv 0$ in $[0, 1]$. Assume $t\in[0, 1/2]$. Then, for any $0<\varepsilon\leqq 1/4$, by the mean-value theorem, there exists a $t_1\in(t+\varepsilon, t+2\varepsilon)$ such that

$$\frac{1}{\varepsilon}(f(t+2\varepsilon) - f(t+\varepsilon)) = f'(t_1). \tag{6.16}$$

Hence, writing $f'(t_1) - f'(t) = \int_t^{t_1} f''(\tau)d\tau$ and noting that $0<t_1-t<2\varepsilon$, we have by (6.16)

$$|f'(t)| \leqq \frac{2}{\varepsilon}\max_I |f(t)| + 2\varepsilon \max_I |f''(t)|.$$

Then, setting $\varepsilon = \frac{1}{4}\{\max_I |f(t)|/(\max_I |f(t)| + \max_I |f''(t)|)\}^{1/2}$ in the above inequality, we have

$$|f'(t)| \leqq 9\{\max_I |f(t)|(\max_I |f(t)| + \max_I |f''(t)|)\}^{1/2}.$$

When $t \in [1/2, 1]$, we can show the inequality in the same way. Hence (6.15) is proved. ∎

THEOREM 6.6 Let $\{a_j\}_{j=1}^{\infty}$ be a bounded set of $\mathscr{A}$. Assume that there exists an $a \in \mathscr{A}$ such that

$$a_j(\eta, y) \to a(\eta, y) \quad \text{in} \quad \mathbf{R}^{2n}_{\eta,y} \qquad (j \to \infty) \tag{6.17}$$

uniformly on any compact set. Then we have

$$\lim_{j\to\infty} \mathrm{O_s}[e^{-iy\cdot\eta}a_j] = \mathrm{O_s}[e^{-iy\cdot\eta}a]. \tag{6.18}$$

Remark This theorem says that $\mathrm{O_s} - \iint$ and limit commute; it corresponds to the Lebesgue dominated convergence theorem. The boundedness of $\{a_j\}_{j=1}^{\infty}$ in $\mathscr{A}$ corresponds to the L_1-boundedness in the dominated convergence theorem.

Proof Set $f_j(t) = a_j(\eta + te_k, y) - a(\eta + te_k, y)$ $(e_k = (0, \dots, \overset{(k)}{1}, \dots, 0))$. Since $\max_I |f_j(t)| \to 0$ and $\{\max_I |f_j(t)| + \max_I |f_j''(t)|\}_{j=1}^{\infty}$ is bounded for any fixed y and η, therefore by Lemma 6.5 we have

$$f_j'(0) = \partial_{\eta_k} a_j(\eta, y) - \partial_{\eta_k} a(\eta, y) \to 0 \quad \text{in} \quad \mathbf{R}^{2n}_{\eta,y} \qquad (j \to \infty)$$

uniformly on any compact set. Repeating this process, we see that, for any α and β, $\partial_\eta^\alpha \partial_y^\beta a_j(\eta, y)$ converges to $\partial_\eta^\alpha \partial_y^\beta a(\eta, y)$ in $\mathbf{R}^{2n}_{\eta,y}$ as $j \to \infty$ uniformly on any compact set. Since a, a_j $(j = 1, 2, \dots)$ are bounded in some $\mathscr{A}^m_{\delta,\tau}$, using (6.9) for a, a_j we get (6.18) by the Lebesgue dominated convergence theorem. ∎

THEOREM 6.7 Let $a \in \mathscr{A}$. Then, for any α, β we have

$$\begin{aligned}\mathrm{O_s}[e^{-iy\cdot\eta}y^\alpha a] &= \mathrm{O_s}[(-D_\eta)^\alpha e^{-iy\cdot\eta}\cdot a] = \mathrm{O_s}[e^{-iy\cdot\eta}D_\eta^\alpha a],\\ \mathrm{O_s}[e^{-iy\cdot\eta}\eta^\beta a] &= \mathrm{O_s}[(-D_y)^\beta e^{-iy\cdot\eta}\cdot a] = \mathrm{O_s}[e^{-iy\cdot\eta}D_y^\beta a].\end{aligned} \tag{6.19}$$

Proof We write $y^\alpha e^{-iy\cdot\eta} = (-D_\eta)^\alpha e^{-iy\cdot\eta}$. Then an integration by parts gives

$$\begin{aligned}\mathrm{O_s}[e^{-iy\cdot\eta}y^\alpha a] &= \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}D_\eta^\alpha(\chi(\varepsilon\eta, \varepsilon y)a(\eta, y))dy\,đ\eta\\ &= \lim_{\varepsilon\to 0}\mathrm{O_s}[e^{-iy\cdot\eta}D_\eta^\alpha(\chi(\varepsilon\eta, \varepsilon y)a(\eta, y))].\end{aligned}$$

If $a \in \mathscr{A}^m_{\delta,\tau}$, we see that $\{D_\eta^\alpha(\chi(\varepsilon\eta, \varepsilon y)a(\eta, y))\}_{0<\varepsilon<1}$ is bounded in $\mathscr{A}^m_{\delta,\tau}$ and converges to $D_\eta^\alpha a(\eta, y)$ in $\mathbf{R}^{2n}_{\eta,y}$ as $\varepsilon \to 0$ uniformly on any compact set. Hence by Theorem 6.6 we get $\mathrm{O_s}[e^{-iy\cdot\eta}y^\alpha a] = \mathrm{O_s}[e^{-iy\cdot\eta}D_\eta^\alpha a]$. For $\mathrm{O_s}[e^{-iy\cdot\eta}\eta^\beta a]$, since $\{D_y^\beta(\chi(\varepsilon\eta, \varepsilon y)a(\eta, y))\}_{0<\varepsilon<1}$ is bounded in $\mathscr{A}^{m+\delta|\beta|}_{\delta,\tau,}$, we can proceed in a similar way. ∎

THEOREM 6.8 For $a \in \mathscr{A}$, we have the following invariance of the oscillatory integral with respect to translations

$$\mathrm{O_s}[e^{-i(y+y_0)(\eta+\eta_0)}a(\eta + \eta_0, y + y_0)] = \mathrm{O_s}[e^{-iy\cdot\eta}a(\eta, y)] \tag{6.20}$$

and for non-negative integers l, l' we have

$$\mathrm{O_s}[e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}a(\eta, y)\}] = \mathrm{O_s}[e^{-iy\cdot\eta}a(\eta, y)]. \tag{6.21}$$

Proof Set $a_0(\eta, y) = e^{-i(y\cdot\eta_0 + y_0\cdot\eta + y_0\cdot\eta_0)}a(\eta + \eta_0, y + y_0)$. Then we have $e^{-i(y+y_0)(\eta+\eta_0)}a(\eta + \eta_0, y + y_0) = e^{-iy\cdot\eta}a_0(\eta, y)$. Hence, using the elementary inequalities

$$\langle\eta + \eta_0\rangle \leqq 2\langle\eta_0\rangle\langle\eta\rangle, \quad \langle y + y_0\rangle \leqq 2\langle y_0\rangle\langle y\rangle, \tag{6.22}$$

we see that $a_0(\eta, y) \in \mathscr{A}$. So, by Definition (6.4) we can write

$$\begin{aligned}\text{left side of (6.20)} &= \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon(\eta - \eta_0), \varepsilon(y - y_0))a(\eta, y)dy\,đ\eta\\ &= \lim_{\varepsilon\to 0}\mathrm{O_s}[e^{-iy\cdot\eta}\chi(\varepsilon(\eta - \eta_0), \varepsilon(y - y_0))a(\eta, y)].\end{aligned}$$

Then by Theorem 6.6 we obtain (6.20). On the other hand, if we use (6.19), then clearly (6.21) holds. ∎

THEOREM 6.9 (1) Let $a(\eta, y) \in \mathscr{A}^m_{\delta,\tau}$ and assume that there exists $(0 \leqq) f(y) \in L_1(\mathbf{R}^n_y)$ such that $|a(\eta, y)| \leqq \langle\eta\rangle^m f(y)$ in $\mathbf{R}^{2n}_{\eta,y}$ holds. Then for $\chi \in \mathscr{S}$ in $\mathbf{R}^n_\eta$ such that $\chi(0) = 1$ we have

$$\mathrm{O_s}[e^{-iy\cdot\eta}a] = \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon\eta)a(\eta, y)dy\,đ\eta. \tag{6.23}$$

When there exists $(0\leqq)\ g(\eta)\in L_1(\mathbf{R}^n_\eta)$ such that $|a(\eta, y)| \leqq g(\eta)\langle y\rangle^\tau$ in $\mathbf{R}^{2n}_{\eta,y}$ holds, we have

$$\mathrm{O}_S[e^{-iy\cdot\eta}a] = \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon y)a(\eta, y)dy\bar{d}\eta. \tag{6.24}$$

(2) Furthermore, when $\int e^{-iy\cdot\eta}a(\eta, y)dy$ and $\int e^{-iy\cdot\eta}a(\eta, y)\bar{d}\eta$ are integrable with respect to η and y, respectively, we can write

$$\begin{aligned}\mathrm{O}_S[e^{-iy\cdot\eta}a] &= \int\left\{\int e^{-iy\cdot\eta}a(\eta, y)dy\right\}\bar{d}\eta,\\ \mathrm{O}_S[e^{-iy\cdot\eta}a] &= \int\left\{\int e^{-iy\cdot\eta}a(\eta, y)\bar{d}\eta\right\}dy.\end{aligned} \tag{6.25}$$

Proof (1) Since $\chi(\varepsilon\eta)a(\eta, y)\in L_1(\mathbf{R}^{2n}_{\eta,y})$ the right-hand side of (6.23) can be written as $\lim_{\varepsilon\to 0}\mathrm{O}_S[e^{-iy\cdot\eta}\chi(\varepsilon\eta)a(\eta, y)]$. Noting that $\{\chi(\varepsilon\eta)a(\eta, y)\}_{0<\varepsilon<1}$ is a bounded set in $\mathscr{A}^m_{\delta,\tau}$, by Theorem 6.6 we get (6.23) and similarly we get (6.24).

(2) We write (6.23) as

$$\mathrm{O}_S[e^{-iy\cdot\eta}a] = \lim_{\varepsilon\to 0}\int\chi(\varepsilon\eta)\left\{\int e^{-iy\cdot\eta}a(\eta, y)dy\right\}\bar{d}\eta.$$

Then, by the Lebesgue dominated convergence theorem, we obtain the upper half of (6.25); the lower half is obtained in a similar way. ∎

Example For $a(x)\in\mathscr{B}(\mathbf{R}^n_x)$, let us calculate $\overline{\mathscr{F}}\mathscr{F}[a]$ in the sense of an oscillatory integral. First, formally we have

$$\overline{\mathscr{F}}\mathscr{F}[a](x) = \iint e^{i(x-y)\cdot\eta}a(y)dy\bar{d}\eta.$$

Then, we have $e^{ix\cdot\eta}a(y)\in\mathscr{A}^0_{0,0}$ with x as a parameter. So we define $\overline{\mathscr{F}}\mathscr{F}[a](x) = \mathrm{O}_S[e^{-iy\cdot\eta}(e^{ix\cdot\eta}a(y))]$. Taking $\chi(\eta)\in\mathscr{S}$ in $\mathbf{R}^n_\eta$ such that $\chi(0)=1$ and setting $\chi(\eta, y) = \chi(\eta)\chi(y)$, we have

$$\begin{aligned}\overline{\mathscr{F}}\mathscr{F}[a](x) &= \lim_{\varepsilon\to 0}\int\chi(\varepsilon y)a(y)\left\{\int e^{i(x-y)\cdot\eta}\chi(\varepsilon\eta)\bar{d}\eta\right\}dy\\ &= \lim_{\varepsilon\to 0}\varepsilon^{-n}\int\chi(\varepsilon y)a(y)\overline{\mathscr{F}}[\chi]\left(\frac{x-y}{\varepsilon}\right)dy\end{aligned}$$

$$\begin{aligned}&= \lim_{\varepsilon\to 0}\int\chi(\varepsilon(x-\varepsilon y'))a(x-\varepsilon y')\overline{\mathscr{F}}[\chi](y')dy'\\ &= a(x)\int\overline{\mathscr{F}}[\chi](y')dy' = a(x)\chi(0) = a(x).\end{aligned} \tag{6.26}$$

Hence we get $\overline{\mathscr{F}}\mathscr{F} = I$ on $\mathscr{B}(\mathbf{R}^n_x)$.

Addition If we replace the number τ in $\mathscr{A}^m_{\delta,\tau}$ by a sequence $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots)$, $\tau_0 \leqq \tau_1 \leqq \cdots$, and define $\mathscr{A}^m_{\delta,\boldsymbol{\tau}}$ as the set of C^∞-functions $a(\eta, y)$ satisfying

$$|\partial^\alpha_\eta\partial^\beta_y a(\eta, y)| \leqq C_{\alpha,\beta}\langle\eta\rangle^{m+\delta|\beta|}\langle y\rangle^{\tau_{|\alpha|}}, \tag{6.27}$$

then the oscillatory integral (6.4) is well-defined for any function $a(\eta, y)\in\mathscr{A}^m_{\delta,\boldsymbol{\tau}}$, and for l, l' such that

$$-2l(1-\delta) + m < -n, \quad -2l' + \tau_{2l} < -n, \tag{6.7'}$$

the value of $\mathrm{O}_S[e^{-iy\cdot\eta}a(\eta, y)]$ is determined by (6.9) (see Kumano-go [5]). In this case for $a(x)\in\mathscr{O}_M$ we obtain $\overline{\mathscr{F}}\mathscr{F}[a](x) = a(x)$, where $\mathscr{O}_M$ denotes the set of moderately increasing functions in Schwartz [1], II, p. 99.