Existence theorem for ODEs:

Let $0\in I\subset\mathbb{R}$ an open intervall. D a finite-dimensional normed vector space over $\mathbb{R}$ or $\mathbb{C}$ and $F: U\subseteq D^k\times I\to D$. We are going to consider local existence and uniqueness of solutions $u: I\to D$ to the ordinary differential equation

$$(*)\qquad \forall t\in I:\quad \partial_t^k u(t) = F(u(t), \partial_t u(t), \dots, \partial_t^{k-1} u(t), t)$$

satisfying the initial conditions

$$u(0) = u_0\in D, \dots, \partial_t^{k-1} u(0) = u_{k-1,0}\in D.$$

Replacing D by D^k and u by $\tilde{u}: I\to D^k\times\mathbb{R}$

$$\tilde{u}(t) := (u(t), \dots, \partial_t^{k-1} u(t), t)$$

$(*)$ is equivalent to $\partial_t\tilde{u}(t) = \tilde{F}(\tilde{u}(t))$ with $\tilde{F}(u_0, \dots, u_{k-1}, s) := (F(u_0, \dots, u_{k-1}, s), 1)$. The initial condition transforms into $\tilde{u}(0) = (u_0, \dots, u_{k-1,0}, 0)$.

So it's sufficient to study

$$(**)\qquad \partial_t u(t) = F(u(t))\ \forall t\in I,\qquad u(0) = u_0.$$

Example: 1.) $\partial_t u = 2\sqrt{u}$, $u(0) = 0$ has solutions $u_\varepsilon(t) = \begin{cases} 0, & t<\varepsilon \\ (t-\varepsilon)^2, & t\geq\varepsilon\end{cases}$

which are all $C^1(\mathbb{R})$, so we cannot in general expect a unique solution.

2.) $\begin{array}{l} \partial_t u = u^2 \\ u(0) = u_0 \end{array}$ has the solution $u(t) = \frac{u_0}{1-u_0 t}$,

which only exists as long as $1 - u_0 t \neq 0$. So we cannot expect to find a solution on arbitrarily large intervals I.

3.) If $\partial_t u = f(t)$, $f \notin C^0$, then $\nexists$ C^1-solution.

Types of Solutions:

Def: A function $u: I \to D$ is said to be a

- classical solution of $(**)$ if $u \in C^1(I, D)$ and $(**)$ is fulfilled.
- strong solution of $(**)$ if $u \in C^0(I, D)$ and $u(t) = u_0 + \int_0^t F(u(s))\, ds$,
- weak solution of $(**)$ if $u \in L^\infty(I, D)$ and $\forall \varphi \in C_c^\infty(I)$:

$$\int_I u(t)\varphi(t)\, dt = u_0 \int_I \varphi(t) + \int_I \varphi(t) \int_0^t F(u(s))\, ds\, dt$$

(i.e. $(**)$ is satisfied in the distributional sense).

Lemma: Let $F \in C^0(D, D)$. Then the three notions of solution are equivalent. In particular, every weak solution is C^1.

Proof: By the Fundamental Theorem of Calculus, every classical Solution is strong. Multiplying the equation fulfilled by a strong solution with φ and integrating, every strong solution is a weak solution. If $u \in L^\infty$ is a weak solution, then $F(u) \in L^\infty$. Thus $\int_0^t F(u(s))\,ds$ is Lipschitz-continuous. (with Lipschitz-constant $\|F(u)\|_\infty$).

As $\partial_t\left(u - \int_0^t F(u)\right) = 0$ in the distributional sense $\overset{\text{Diff.-Int.}}{\Rightarrow} u(t) = u_0 + \int_0^t F(u(s))\,ds$, so also u is Lipschitz-continuous. But then $F(u)$ is continuous, so $\int_0^t F(u(s))\,ds \in C^1$, and hence $u(t) \in C^1$ with $\partial_t u = F(u)$. $\square$

<u>Theorem:</u> (Picard-Lindelöf)

Let $\emptyset \neq \Omega \subset \mathcal{D}$, $N_\varepsilon(\Omega) := \{ u \in \mathcal{D} : \|u - v\|_{\mathcal{D}} < \varepsilon$ for some $v \in \Omega \}$ the ε-neighborhood of Ω and $F : \overline{N_\varepsilon(\Omega)} \to \mathcal{D}$ Lipschitz with Lipschitz constant $M > 0$. Given $0 < T < \min\left\{ \frac{\varepsilon}{\|F\|_\infty}, \frac{1}{M} \right\}$, let $I = (-T, T)$. Then for every $u_0 \in \Omega$ $\exists!$ classical solution $u : I \to \overline{N_\varepsilon(\Omega)}$ of

$$\partial_t u(t) = F(u(t)) \quad \forall t \in I$$
$$u(0) = u_0$$

The solution maps $\begin{aligned} S(t) : \Omega &\to \mathcal{D} \\ u_0 &\mapsto u(t) \end{aligned}$ and $\begin{aligned} S : \Omega &\to C^0(\overline{I}, \mathcal{D}) \\ u_0 &\mapsto u \end{aligned}$ are Lipschitz-continuous with Lipschitz constant $\leq \frac{1}{1 - TM}$.

<u>Remark:</u> If $F \in C^k \Rightarrow S, S(t) \in C^k$, $u \in C^{k+1}(I, \mathcal{D})$ (Exercise!)

<u>Proof:</u> Write $\Omega_\varepsilon := \overline{N_\varepsilon(\Omega)}$, and define (for $u_0 \in \Omega$)

$$\Phi_{u_0} : C^0(\overline{I}, \Omega_\varepsilon) \to C^0(\overline{I}, \Omega_\varepsilon)$$
$$u \mapsto \left(t \mapsto u_0 + \int_0^t F(u(s))\,ds\right)$$

By definition, a strong solution is nothing but a fixed point of Φ_{u_0}. As strong solutions are classical, we only have to find a fixed point of Φ_{u_0}.

Note Φ_{u_0} indeed maps $C^0(\bar{I},\Omega_\varepsilon) \longrightarrow C^0(\bar{I},\Omega_\varepsilon)$:

$t \longmapsto u_0 + \int_0^t F(u(s))\,ds$ is continuous and the right hand side

$\in \Omega_\varepsilon$: $\| \int_0^t F(u(s))\,ds \|_\infty \leq T \|F\|_\infty < \frac{\varepsilon}{\|F\|_\infty} \|F\|_\infty = \varepsilon$.

As F is Lipschitz,

$$\| \Phi_{u_0}(u)(t) - \Phi_{u_0}(v)(t) \|_{\mathcal{D}} = \| \int_0^t ds \, (F(u(s)) - F(v(s))) \|_{\mathcal{D}}$$
$$\leq \int_0^t M \, \| u(s) - v(s) \|_{\mathcal{D}} \, ds$$
$$\leq \underbrace{TM}_{<1} \, \| u - v \|_{C^0(\bar{I},\mathcal{D})} .$$

Therefore Φ_{u_0} is a strict contraction on the complete metric space $C^0(\bar{I},\Omega_\varepsilon)$ with contraction constant <1.
The contraction mapping theorem (see below) implies that Φ_{u_0} has a fixed point $u = \Phi_{u_0}(u)$, which is unique.

Let $u_0, \tilde{u}_0$ two initial data in Ω, $u, \tilde{u}$ the corresponding solutions. By construction $\Phi_{u_0}(u) = u$ and $\Phi_{u_0}(\tilde{u}) = \Phi_{\tilde{u}_0}(\tilde{u}) + u_0 - \tilde{u}_0 = \tilde{u} + u_0 - \tilde{u}_0$, and thus

$$u - \tilde{u} = \Phi_{u_0}(u) - \Phi_{u_0}(\tilde{u}) + u_0 - \tilde{u}_0 .$$

$$\Rightarrow \quad \| u - \tilde{u} \|_{C^0(\bar{I},\mathcal{D})} \leq \underbrace{TM}_{<1} \, \| u - \tilde{u} \|_{C^0(\bar{I},\mathcal{D})} + \| u_0 - \tilde{u}_0 \|_{\mathcal{D}}$$

$\Rightarrow \quad \| u - \tilde{u} \|_{C^0(\bar{I},\mathcal{D})} \leq \frac{1}{1-TM} \| u - \tilde{u}_0 \|_{\mathcal{D}} \Rightarrow$ Lipschitz property of S and $S(t)$. $\square$

In the proof we have used the <u>contraction mapping theorem</u>:

<u>Exercise</u>: Let (X,d) complete, nonempty metric space, $\phi : X \to X$ a strict contraction on X, thus there exists $0<c<1$ $\forall u,v \in X$: $d(\phi(u),\phi(v)) \leq c\, d(u,v)$. Show that ϕ has a unique fixed point $u = \phi(u)$.

<u>Hint</u>: Let $u_0 \in X$ arbitrary. Show that $u_{n+1} := \phi(u_n)$ defines a Cauchy sequence: Indeed $\forall \varepsilon>0$ $\forall n \geq m > \log\left(\frac{\varepsilon(1-c)}{d(u_1,u_0)}\right) / \log(c)$

$$d(u_n,u_m) \leq \underbrace{d(u_n,u_{n-1})}_{\leq c^{n-m+1} d(u_{m+1},u_m)} + \dots + \underbrace{d(u_{m+2},u_{m+1})}_{\leq c\, d(u_{m+1},u_m)} + d(u_{m+1},u_m)$$

$$\leq (1+c+c^2+\dots)\, d(u_{m+1},u_m) \leq \frac{1}{1-c} c^m d(u_1,u_0) < \varepsilon$$

As X is complete, u_n converges to $u \in X$, and $u = \phi(u)$.

Concerning uniqueness (it actually also follows from the above proof):

<u>Lemma</u>: (Gronwall's inequality)

Let $u : [t_0,t_1] \to [0,\infty)$ continuous s.t. $u(t) \leq A + \int_{t_0}^t B(s)u(s)\,ds$ for all $t \in [t_0,t_1]$, where $A \geq 0$, $B : [t_0,t_1] \to [0,\infty)$ continuous.

$$\Longrightarrow \quad u(t) \leq A \exp\left(\int_{t_0}^t B(s)\,ds\right) \qquad \forall t \in [t_0,t_1].$$

<u>Proof</u>: By a limiting argument it is enough to prove the claim for $A>0$.

By assumption and the Fundamental Theorem of Calculus

$$\frac{d}{dt}\left(A + \int_{t_0}^t B(s)u(s)\,ds\right) \leq B(t)\left(A + \int_{t_0}^t B(s)u(s)\,ds\right).$$

Chain rule: $\frac{d}{dt} \log\left(A + \int_{t_0}^t B(s)u(s)\,ds\right) \leq B(t)$.

Fundament Theorem of Calculus: $\log\left(A + \int_{t_0}^t B(s)u(s)\,ds\right) \leq \log(A) + \int_{t_0}^t B(s)\,ds$

Exponentiate and apply the assumption again. $\boxed{\text{q}}$

Thm: (Uniqueness)

Suppose $u, v \in C^1(I, D)$ are classical solutions to $\partial_t u(t) = F(u(t))$, F locally Lipschitz. If $u(t_0) = v(t_0)$ for some $t_0 \in I \Rightarrow u = v$.

Proof: Wlog I compact (otherwise write $I = \bigcup_n I_n$, I_n compact), and $t_0 = 0$. We show $u(t) = v(t)$ $\forall t \in I, t \geq 0$. $t \leq 0$ similar.

By assumption,

$$\partial_t (u(t) - v(t)) = F(u(t)) - F(v(t))$$

and by the Fundamental Theorem of Calculus

$$u(t) - v(t) = \underbrace{u_0 - v_0}_{=0} + \int_0^t \left(F(u(s)) - F(v(s))\right) ds.$$

So $$\|u(t) - v(t)\|_D \leq \int_0^t \underbrace{\|F(u(s)) - F(v(s))\|_D}_{\leq M \|u(s) - v(s)\|_D} ds \quad \forall t \in [0,T]$$

Gronwall's inequality with $u := \|u(t) - v(t)\|_D$, $A = 0$, $B = M$

$\Rightarrow$ assertion $\square$

Remark: • The Peano Existence Theorem assures the existence of solutions if F is merely continuous. Our above example shows that uniqueness does not hold for merely continuous right hand sides.

• However, carrying the t-dependence along in the above proofs, one can show that there exists a unique solution of

$$\partial_t u = f(t, u), \quad u(0) = u_0 \quad \text{if } f \text{ is continuous in } t \text{ and Lipschitz in } u.$$

• Unlike for ODE, there are natural PDE without any local solutions:

Lewy's example: $\partial_{x_1} u + i \partial_{x_2} u + i(x_1 + i x_2) \partial_{x_3} u = f \quad (*)$

$\exists f \in C^\infty(\mathbb{R}^3)$ $\forall \Omega \subset \mathbb{R}^3$ open $\nexists u \in \mathcal{D}'(\Omega)$ s.t. $(*)$ is satisfied in $\mathcal{D}'(\Omega)$.

Upshot: Given (x_0, ξ_0) and $p(x, \xi)$ smooth $\Rightarrow \exists!$ integral curve of H_p through (x_0, ξ_0).

Linear equations: If $F(u) = Au + b$, Gronwall's inequality implies that $|u(t)| \leq C e^{Dt}$ for any t in the intervall of existence. Let $(-T_0, T_1)$ the maximal intervall of existence. We are going to see that $T_0 = T_1 = \infty$, and restrict ourselves to T_1.

Indeed, $|u(t)| \leq C e^{DT_1}$ for $t \geq 0$, and F is Lipschitz continuous with Lipschitz constant $\|A\|$. Let $u_0 = u(T_1 - \frac{1}{2\|A\|})$

$$\Rightarrow \quad \begin{array}{l} \partial_t u = F(u) \\ u(0) = u_0 \end{array} \quad \text{has a solution } \tilde{u} : (-T, T) \to D = N_\infty(\Omega)$$

$$\text{if } T < \frac{1}{\|A\|}.$$

Defining $u_{new}(t) := \begin{cases} u(t) & , \ t \in (-T_0, T_1) \\ \tilde{u}(t - T_1 + \frac{1}{2\|A\|}) & , \ t \in (T_1 - \frac{1}{2\|A\|}, T_1 + \frac{1}{2\|A\|}) \end{cases}$

extends u to a solution beyond T_1. So $(-T_0, T_1)$ was not a maximal interval of existence $\lightning$ $\Rightarrow T_1 = \infty$.

Remark: The proof readily generalizes to nonlinearities $|F(u)| \leq C(1 + |u|)$.

ODEs can be used to solve first-order PDEs ("quasilinear" ones, at lea
We will have to solve "transport equations" of the form

$$(***) \begin{cases} \partial_t u(t,x) + \sum_{j=1}^{n} a_j(t,x)\, \partial_{x_j} u(t,x) = f(t,x) \\ u(0,x) = u_0(x) \quad \forall x \end{cases} , \quad a_j, f, u_0 \text{ smooth}$$

Picard-Lindelöf $\Rightarrow$ $\exists!$ solution of the ODE (for small t)

$$\partial_t x_j(t) = a_j(t, x(t)) \quad , \quad x_j(0) = x_j^0$$

$$(((\partial_t t = 1 \qquad\qquad t(0) = 0)))$$

We write $X(t) = \mathcal{F}_t^a x^0$, the time-$t$ flow of x^0 under the vector field a.

Furthermore $\frac{d}{dt} u(t, x(t)) = \partial_t u + \sum_j a_j(t, x(t)) \partial_{x_j} u = f(t, x(t))$

$u(0, x(0)) = u_0(x^0)$.

Vector fields and flows

Let $\Omega \subset \mathbb{R}^n$ open, V vector field, i.e. $V: \Omega \to \mathbb{R}^n$.

Def: The flow of V is the map $\underset{\substack{\cap \\ (-\varepsilon,\varepsilon)}}{(t}, \underset{\substack{\cap \\ \mathbb{R}^n}}{x)} \mapsto \underset{}{\mathcal{F}_t x} \in \mathbb{R}^n$, where $\mathcal{F}_t x$ is the solution of the ODE
$$\begin{cases} \frac{d}{dt}(\mathcal{F}_t x) = V(\mathcal{F}_t x) \\ \mathcal{F}_0 x = x \end{cases}.$$

If V is Lipschitz, Picard-Lindelöf asserts that $\forall x \in \Omega$ there exists a C^1 solution to this ODE in a small interval around 0, and the solution depends Lipschitz-continuously on x. If x varies over a compact subset of Ω, we may chose a common interval of existence for the solution.
In the following we are going to assume $V \in C^1$. Then also the x-dependence is C^1.

Remark: The map $t \mapsto \mathcal{F}_t x$ is an integral curve for V.

Theorem: Let S be a hypersurface in $\mathbb{R}^n$, V a vector field on an open neighborhood of S, and V nowhere tangential to S.

$\Longrightarrow$ There exist an open neighborhood U of $\{0\} \times S$ in $\mathbb{R} \times S$ and an open neighborhood V of $S \subset \mathbb{R}^n$ such that

$(t,x) \mapsto \mathcal{F}_x x$ is a diffeomorphism from U onto V.

Proof: We write S locally as a graph of $g: A \subseteq \mathbb{R}^{n-1} \to \mathbb{R}^n$ and apply the Inverse Function Theorem:

The derivative of $(t,a) \mapsto \mathcal{F}_t g(a)$ in the point $(0, g(a))$ is

$$\left(\partial_t \mathcal{F}_t g(a), \partial_{a_1} \mathcal{F}_t g(a), \dots, \partial_{a_{n-1}} \mathcal{F}_t g(a)\right)\Big|_{t=0} = \left(V(g(a)), \partial_{a_1} g(a), \dots, \partial_{a_{n-1}} g(a)\right).$$

Because $\partial_{a_1} g(a), \dots, \partial_{a_{n-1}} g(a)$ span the tangent space of S in $g(a)$ and $V(g(a))$ is not tangent to S in $g(a)$, the columns of this matrix are linearly independent, i.e. the derivative of $(t,a) \mapsto \mathcal{F}_t g(a)$ is invertible in $(0, g(a))$. By the inverse function theorem $(t,a) \mapsto \mathcal{F}_t g(a)$ is a local diffeomorphism.

Covering S by coordinate charts, we obtain a surjective differentiable map. Possibly restricting to a smaller neighborhood of $\{0\} \times S$, we obtain injectivity. $\boxtimes$

<u>Theorem:</u> Let $a_j, b, f \in C^1$, a_j, b real valued. Consider the PDE $(*)$ $Lu(x) := \sum_{j=1}^{n} a_j(x) \partial_{x_j} u(x) + b(x) u(x) = f(x)$ and a hypersurface $S \subset \mathbb{R}^n$ such that $A(x) := (a_1(x), \dots, a_n(x))$ is nowhere tangent to S.

$\Longrightarrow$ $\forall \varphi \in C^1(S)$ there exists a unique C^1-solution u of

$Lu = f$, $u|_S = \varphi$ defined in a small neighborhood of S.

<u>Proof:</u> Let $x(t) := \Phi_t x$ be the flow associated to the vector field A.

<u>Uniqueness:</u> Let u_1, u_2 be two solutions to $Lu = f$, $u|_S = \varphi$.

$\longrightarrow$ $\vartheta(t) := u_1(x(t)) - u_2(x(t))$ satisfies

$$(**) \quad \partial_t \vartheta(t) = \sum_{j=1}^{n} \partial_{x_j} \big(u_1(x(t)) - u_2(x(t))\big) \, \partial_t x_j(t)$$
$$= \sum_{j=1}^{n} a_j(x(t)) \, \partial_{x_j} \big(u_1(x(t)) - u_2(x(t))\big)$$
$$= f(x(t)) - b(x(t)) u_1(x(t)) - f(x(t)) + b(x(t)) u_2(x(t))$$
$$= -b(x(t)) \vartheta(t)$$

and $\vartheta(0) = u_1(x(0)) - u_2(x(0)) = \varphi(x(0)) - \varphi(x(0)) = 0$.

Picard-Lindelöf: $\vartheta(t) \equiv 0$ is the unique solution of this ODE

$\Longrightarrow$ $u_1(x(t)) = u_2(x(t))$.

By the previous theorem, every point in a neighborhood of S can be written as $x(t)$ for some x, t $\Longrightarrow$ $u_1 \equiv u_2$ near S.

<u>Existence:</u> Analogous: Solve the initial value problem

$$\partial_t x(t) = A(x(t)) \qquad x(0) = x_0 \in S$$
$$\partial_t \vartheta(t) = f(x(t)) + b(x(t)) \vartheta(t) \qquad \vartheta(0) = \varphi(x_0)$$

to get C^1-functions $x(t, x_0)$, $\vartheta(t, x_0)$. Computation like in $(**)$ shows that $u(x(t, x_0)) := \vartheta(t, x_0)$ satisfies $(*)$.
As $x(t, x_0) = \Phi_t x_0$ local diffeomorphism, we obtain a solution u. $\boxtimes$