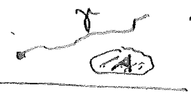
Guest lecture: Morten Risager, Spectral theory for hyperbolic surfaces

$\mathcal{H} = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$, $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$



$\gamma(t)$, $t \in [0,1]$. $\gamma(t) = x(t) + i y(t)$

$$\ell(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt$$

$d\mu(z) = \dfrac{dx\,dy}{y^2}$, $\mu(A) = \int_A \frac{1}{y^2}\, dx\, dy$

$\Delta = -y^2 \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \right)$ Laplace operator

Spectrum of Δ on $\mathbb{H}$ can be completely described, it's boring.

$\gamma \in SL_2(\mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a,b,c,d \in \mathbb{R}\ ad-bc = 1 \}$, $T_\gamma : \mathcal{H} \to \mathcal{H}$, $z \mapsto \dfrac{az+b}{cz+d}$

$\operatorname{Im}(T_\gamma(z)) = \dfrac{\operatorname{Im}(z)}{|cz+d|^2} > 0 \implies T_\gamma(z) \in \mathcal{H}$ indeed.

$\operatorname{Isom}^+(\mathcal{H})$ = orientation preserving isometries of $\mathcal{H}$

$= \{ T_\gamma : \gamma \in SL_2(\mathbb{R}) \} \cong SL_2(\mathbb{R}) / \{\pm 1\} = PSL_2(\mathbb{R})$

Let Γ be a discrete subgroup of $PSL_2(\mathbb{R})$. What is the topology? Consider $SL_2(\mathbb{R})$ as subset of $\mathbb{R}^4$. $PSL_2(\mathbb{R})$ is one connected component (or take the quotient topology).

Example: 1.) $\Gamma_\infty = \{ \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix} : n \in \mathbb{Z} \}$ corresponding to translations $z \mapsto z+n$.

2.) $PSL_2(\mathbb{Z}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) : a,b,c,d \in \mathbb{Z} \} / \{\pm 1\}$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longleftrightarrow z \mapsto z+1$

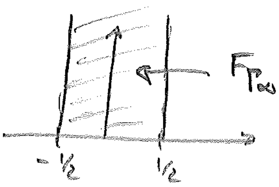
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longleftrightarrow z \mapsto -\frac{1}{z}$ These two transformations generate $PSL_2(\mathbb{Z})$.

Def: A measurable set F_Γ is called a <u>fundamental domain</u> for Γ if

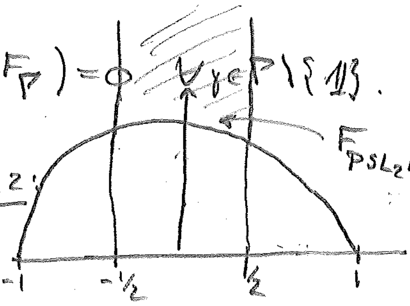
1.) $\bigcup_{\gamma \in \Gamma} \gamma F_\Gamma = \mathcal{H}$.

2.) $\mu(\gamma F_\Gamma \cap F_\Gamma) = 0 \quad \forall \gamma \in \Gamma \setminus \{1\}$.

Example 1:

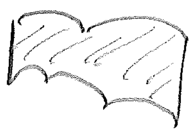


Example 2:



$F_{PSL_2(\mathbb{Z})} = \{ z \in \mathbb{C} : -\frac{1}{2} \le \operatorname{Re} z \le \frac{1}{2},\ |z| \ge 1 \}$

Fact: If Γ is finitely generated $\Rightarrow$ $\exists$ "nice" fundamental domain



Fact: $\mu(F_\Gamma)$ independent of choice of fundamental domain.

Assume $\mu(F_\Gamma) < \infty$ and that "$\mathcal{H}/\Gamma$ has only one cusp" (for simplicity),

i.e. there exists fundamental domain going off to ∞ and not touching the real axis. (If there is no cusp, $\mathbb{H}/\Gamma$ is a compact manifold and the theory is easier - as in the lectures).

Consider
$$D_\Gamma := \Big\{ f : \mathcal{H} \to \mathbb{C} \; : \; f \in C^2,\ f(T_\gamma(z)) = f(z) \ \forall \gamma \in \Gamma, \ \int_{F_\Gamma} \left(|f(z)|^2 + |\Delta f|^2 \right) d\mu(z) < \infty \Big\} = H^2_{loc}(\mathcal{H})^{\overset{\Gamma\text{-invariant fcts}}{\Gamma}} (\cap\, C^2)$$

$$\subseteq L^2(\Gamma) = \Big\{ f : \mathcal{H} \to \mathbb{C} \; : \; f(T_\gamma(z)) = f(z), \ \int_{F_\gamma} |f|^2 < \infty \Big\} = L^2_{loc}(\mathcal{H})^{\Gamma} \leftarrow \Gamma\text{-invariant fcts.}$$

$L^2(\Gamma)$ is a Hilbert space with inner product $\langle f, g \rangle = \int f \bar{g} \, d\mu$.

Theorem: (Δ, D_Γ) is essentially selfadjoint.

Denote by A_Γ the selfadjoint closure ("automorphic Laplacian").

Theorem: The spectrum of A_Γ consists of $\sigma_{disc} = \{ 0 = \lambda_0 < \lambda_1 \le \dots \le \lambda_n \le \dots \}$
$\lambda_n \to \infty$, if there are only finitely many λ_n.

and $\sigma_{cont} = [\frac{1}{4}, \infty)$.

Conjecture 1: (Rölcke-Selberg ~ 1960)

$$\# \{ \lambda \in \sigma_{disc} : \lambda \le X \} \sim \frac{\mu(F_\Gamma)}{4\pi} X \qquad \text{Weyl law}$$

Conjecture 2: (Phillips-Sarnak ~1980)

$\# \{ \lambda \in \sigma_{disc} : \lambda \leq X \}$ is bounded unless Γ is "arithmetic" (like $PSL_2(\mathbb{Z})$)

Conjecture 3: For Γ arithmetic, the eigenvalue multiplicities are bounded.

It has been shown in the 1990s that Conjecture 3 $\Rightarrow$ exceptions to Conjecture 1.

Cuspidal functions:

Consider $C(\Gamma) := \{ f \in L^2(\Gamma) : f \text{ smooth, bounded, } \int_0^1 f(x+iy)\,dx = 0 \}$

(assuming $\Gamma_\infty \subseteq \Gamma$ and that $F_\Gamma \cap \{|z|>R\} = \{z \in \mathbb{C} : |z|>R,\ \operatorname{Re} z \in [-\frac{1}{2},\frac{1}{2}]\}$ for large enough R)

Then we can use Fourier series: $f(z+1) = f(z)$ since $\Gamma_\infty \subseteq \Gamma$:

$$f(x+iy) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a_n(y)\, e^{2\pi i n x}$$, zero Fourier coeff. vanishing,

$$\Delta f = \Delta \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a_n(y)\, e^{2\pi i n x} = \sum_{n\neq 0} -y^2\left((2\pi i n)^2 a_n(y) + a_n''(y)\right) e^{2\pi i n x} \in C(\Gamma).$$

Def: An eigenfunction φ_j of A_Γ in $C(\Gamma)$ is called a cusp form: $A_\Gamma \varphi_j = \lambda_j \varphi_j$

Thm: $\operatorname{span}\{\text{cusp forms}\}$ is dense in $C(\Gamma)$ endowed with the L^2-norm.

If $C(\Gamma)$ is infinite dim. $\Rightarrow$ Conjecture 2 wrong.

$f \in C(\Gamma)$, $f = \sum \langle f, \varphi_j \rangle \varphi_j$

Even for $\Gamma = PSL_2(\mathbb{Z})$, one cannot give a single "explicit" cusp form.

(For certain subgroups of $PSL_2(\mathbb{Z})$, there are special constructions),

but there are existence theorems and they can be determined numerically.

Eisenstein series: $\quad \Gamma_\infty \subseteq \Gamma$, $\gamma \in \Gamma_\infty$, $\quad T_\gamma(\infty) := \dfrac{a\infty + b}{0\cdot\infty + d} := \infty$

$\Gamma_\infty = \{\gamma \in \Gamma : \gamma \text{ upper triangular}\}$

Define: $\quad \underset{\mathcal{H}\ \ \mathbb{C}}{E(z,s)} := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma z)^s \quad$ absolutely convergent for $\operatorname{Re} s > 1$.

Theorem: (Selberg)

i) $E(z,s)$ converges for $\operatorname{Re}(s) > 1$, holomorphic

ii) $\Delta E(z,s) = s(1-s) E(z,s)$

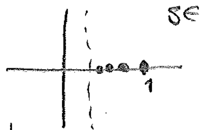
iii) $E(T_\gamma z, s) = E(z,s)$

iv) $E(z,s) \notin L^2(\Gamma)$

$\rightarrow$ v) $E(z,s)$ admits meromorphic continuation to $s \in \mathbb{C}$.

vi) $E(z,s)$ has a pole at $s=1$ with residue $\dfrac{1}{\mu(F_\Gamma)}$

$\rightarrow$ vii) In $\operatorname{Re}(s) \geq \frac{1}{2}$, all poles are real and $\neq \frac{1}{2}$.



viii) $E(z,s)$ admits a Fourier expansion

$$E(z,s) = y^s + \varphi(s) y^{1-s} + \sum_{n \neq 0} \varphi_n(s) \sqrt{y}\, \underset{\text{Bessel fct.}}{K_{s-\frac{1}{2}}}(2\pi |n| y)\, e^{2\pi i n x}$$

where $\varphi(s) = \sqrt{\pi}\, \dfrac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{c>0} \dfrac{1}{c^{2s}} \sum_{d \bmod c} 1 \quad , \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

Γ-fct.

$\rightarrow$ ix) $E(z,s) = \phi(s)\, E(z, 1-s)$

Example: $\Gamma = PSL_2(\mathbb{Z}) \implies \varphi(s) = \sqrt{\pi}\, \dfrac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, \dfrac{\zeta(2s-1)}{\zeta(2s)}$

Riemann's ζ-fct.

(vii) $\implies \zeta(1+it) \neq 0 \quad$ This was the essential ingredient in the prime number theorem.

Theorem: (Spectral expansions)

For $f \in L^2(\Gamma)$ (compactly supported, for simplicity) we have

$$f = \underbrace{\sum_{\substack{\varphi_j \text{ ONB} \\ \text{of cusp forms}}} \langle f, \varphi_j\rangle \varphi_j}_{\text{cusp forms}} + \underbrace{\overset{\text{finite sum}}{\sum_{\substack{u_j,\ u_j \text{ residue} \\ \text{of } E(z,s) \text{ in } (\frac{1}{2},1]}}} \langle f, u_j\rangle u_j}_{L^2 \text{ eigen fcts}} + \underbrace{\frac{1}{4\pi} \int_{-\infty}^{\infty} \langle f, E(z, \tfrac{1}{2}+it)\rangle E(z, \tfrac{1}{2}+it)\, dt}_{\text{Eisenstein series}}$$

$u_j = \operatorname{res}_{s=s_j} E(z,s)$, $\quad \Delta_\Gamma u_j = s_j(1-s_j) u_j$

The three components are mutually orthogonal.

There is a "Weyl law" for such groups:

<u>Thm</u>: (Weyl law)

$$\#\{\lambda \in \sigma_{disc} : \lambda \leq X\} + \frac{1}{4\pi}\int_{-\sqrt{X}}^{\sqrt{X}} -\frac{\varphi'}{\varphi}(\tfrac{1}{2}+it)\,dt$$

$$= \frac{\mu(F_\Gamma)}{4\pi} X + \overset{\sim \# \text{cusps}}{c_1} \sqrt{X}\log(X) + c_2\sqrt{X} + O\left(\frac{\sqrt{X}}{\log(X)}\right)$$

So the conjectures 1-3 essentially ask whether $\frac{1}{4\pi}\int_{-\sqrt{X}}^{\sqrt{X}} \cdots$ is of lower order.

E.g. for $PSL_2(\mathbb{Z})$ this is known (uses Selberg's trace formula relating the geometry with spectral data).

<u>Reference</u>: H. Iwaniec, Introduction to the spectral theory of automorphic forms, 1995.