Perturbations of self-adjoint operators (Kim's talk)

(based on Reed / Simon, Methods of Modern Math. Physics, Vol 2.)

Let H be a Hilbert space, $A : D(A) \subset H \to H$ linear.
$B : D(B) \subset H \to H$

$$(A+B)\varphi := A\varphi + B\varphi \quad \text{for } \varphi \in D(A+B) := D(A) \cap D(B)$$

"A selfadjoint, B symmetric, small $\Longrightarrow$ A+B selfadjoint"

<u>Def</u>: A, B densely defined s.t.

- $D(B) \supset D(A)$
- $\exists a, b \geq 0 \; \forall \varphi \in D(A) : \|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\|$ (1)

Then B is said to be <u>(a,b)-bounded wrt A</u>.

Note that (1) $\Rightarrow$ $\|B\varphi\|^2 \leq 2a^2 \|A\varphi\|^2 + 2b^2 \|\varphi\|^2$

and $\|B\varphi\|^2 \leq a' \|A\varphi\|^2 + b' \|\varphi\|^2 \Rightarrow$ (1) w/ $a = \sqrt{a'}$, $b = \sqrt{b'}$.

<u>Thm</u>: (<u>Kato-Rellich</u>) A selfadjoint, B symmetric s.t. B is (a,b)-bounded wrt. A for some $a < 1$, $b \geq 0$. $\Longrightarrow$
$A+B$ selfadjoint on $D(A)$ and essentially selfadjoint on any core of A.
Moreover, if A is bounded below by M $\Rightarrow$ $A+B$ bounded below by $M - \max\{\frac{b}{1-a}; a|M| + b\}$.

<u>Proof</u>: <u>selfadjointness on $D(A)$</u>: $A+B$ symmetric on the dense domain $D(A)$, so by Grubb, Thm 12.10, (12.13), we need to show $\operatorname{Ran}(A+B \pm i\mu) = H$ for some $\mu > 0$.

<u>"+"</u>: $(A+B+i\mu)\varphi = (\mathbb{1} + B\underbrace{(A+i\mu)^{-1}}_{\text{exists by Grubb, Thm 12.10 } \forall \mu > 0})\overbrace{(A+i\mu)}^{\operatorname{Ran}(A+i\mu) = H \text{ by Grubb, Thm 12.10 } \forall \mu \geq 0}\varphi$ for $\varphi \in D(A)$.

So to conclude $\operatorname{Ran} A+B+i\mu = H$, we only have to show that $\mathbb{1} + B(A+i\mu)^{-1}$ is invertible for some $\mu > 0$.

Note that $\forall \varphi \in D(A)$: $\|(A+i\mu)\varphi\|^2 = \|A\varphi\|^2 + \mu^2\|\varphi\|^2$

$\varphi = (A+i\mu)^{-1}\psi$, $\psi \in H$: $\forall \psi \in H$:

$$\|\psi\|^2 = \|(A+i\mu)(A+i\mu)^{-1}\psi\|^2 = \|A(A+i\mu)^{-1}\psi\|^2 + \mu^2\|(A+i\mu)^{-1}\psi\|^2$$

So $\|A(A+i\mu)^{-1}\psi\|^2 = \|\psi\|^2 \underbrace{-\mu^2\|(A+i\mu)^{-1}\psi\|^2}_{\leq 0} \leq \|\psi\|^2$

$$\|(A+i\mu)^{-1}\psi\|^2 = \frac{1}{\mu^2}\left(\|\psi\|^2 \underbrace{-\|A(A+i\mu)^{-1}\psi\|^2}_{\leq 0}\right) \leq \frac{\|\psi\|^2}{\mu^2}$$

$$\Rightarrow \|B(A+i\mu)^{-1}\psi\| \leq a\|A(A+i\mu)^{-1}\psi\| + b\|(A+i\mu)^{-1}\psi\| \leq \left(a+\frac{b}{\mu}\right)\|\psi\|$$

As $a<1$, we may choose μ large so that $a+\frac{b}{\mu}<1$, i.e. $\|B(A+i\mu)^{-1}\|<1$.

Von Neumann series: $\mathbb{1} + B(A+i\mu)^{-1}$ is invertible

(inverse: $\sum_{j=0}^{\infty}\left(-B(A+i\mu)^{-1}\right)^j$)

<u>$A+B$ essentially selfadjoint on any core of A:</u>

C core of A, i.e. $C \subset D(A)$, $\overline{A|_C} = \overline{A} \underset{\substack{\uparrow \\ A \text{ selfadj., in particular closed}}}{=} A$

Show $\overline{(A+B)|_C} = A+B$: „$\subseteq$" $(A+B)|_C \subset A+B$ as restriction

$A+B$ closed $\Rightarrow \overline{(A+B)|_C} \subset A+B$.

„$\supseteq$" Let $\psi \in D(A) \rightsquigarrow (\psi, A\psi) \in \mathcal{G}(\overline{A|_C}) = \overline{\mathcal{G}(A|_C)}$, so $\exists (\psi_n) \subseteq C$

$\psi_n \to \psi$, $A\psi_n \to A\psi$..

$$\Rightarrow \|\overline{(A+B)|_C}\psi_n - (A+B)\psi\| \leq (a+1)\|A\psi_n - A\psi\| + b\|\psi_n - \psi\| \to 0$$

$$\Rightarrow (\psi, (A+B)\psi) \in \overline{\mathcal{G}((A+B)|_C)} = \mathcal{G}(\overline{(A+B)|_C}). \Rightarrow \overline{(A+B)|_C} \supseteq A+B.$$

$A \geq M \Rightarrow A+B \geq M - \max\{\frac{b}{1-a}, a|M|+b\}$:

Show $\underbrace{\inf_{\substack{\psi \in D(A) \\ \|\psi\|=1}} \langle \psi, (A+B)\psi \rangle}_{= \inf \sigma(A+B)} \geq M - \max\{\frac{b}{1-a}, a|M|+b\}$, i.e.

Show that $\forall t$ s.t. $-t < M - \max\{\frac{b}{1-a}, a|M|+b\}$, $A+B+t$ invertible.

Note $\inf \sigma(A) \geq M > -t$, so $A+t$ invertible $\Rightarrow$

$$(A+B+t)\varphi = \underbrace{(\mathbb{1} + B(A+t)^{-1})}_{\text{sufficient to show that this is invertible}}(A+t)\varphi \quad \forall \varphi \in D(A).$$

'The argument' for this is analogous to the Neumann-series argument above. $\boxed{}$

<u>Schrödinger operators</u>: Let $d \in \mathbb{N}$.

- $-\Delta$ self-adjoint ≥ 0 on $L^2(\mathbb{R}^d)$, $D(-\Delta) = H^2(\mathbb{R}^d)$.
- Multiplication operator associated to $V: \mathbb{R}^d \to \mathbb{R}$ is selfadjoint with domain $\{\psi \in L^2(\mathbb{R}^d) : V\psi \in L^2(\mathbb{R}^d)\} =: D(V)$.

Consider $d=3$, $V = V_2 + V_\infty \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, $\psi \in H^2(\mathbb{R}^3) \Rightarrow$:

$$\begin{aligned} \|V\psi\|_2 &\leq \|V_2\psi\|_2 + \|V_\infty \psi\|_2 \\ &\leq \|V_2\|_2 \|\psi\|_\infty + \|V_\infty\|_\infty \|\psi\|_2 \\ \text{(Sobolev embedding)} &\leq C(\|V_2\|_2 + \|V_\infty\|_\infty)\|\psi\|_{H^2} \Rightarrow D(V) \supset D(-\Delta). \end{aligned}$$

$$\Rightarrow D(-\Delta + V) = D(-\Delta) = H^2(\mathbb{R}^d)$$

Show that $-\Delta + V$ with this domain is self-adjoint. This follows from:

<u>Lemma</u>: $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \Rightarrow \forall a > 0 \; \exists b > 0 : \|V\psi\|_2 \leq a\|-\Delta\psi\|_2 + b\|\psi\|_2$

Proof: $\forall \psi \in H^2(\mathbb{R}^3)$:

$$\|\psi\|_\infty = (2\pi)^{-3} \sup_{x\in\mathbb{R}^3} \left| \int_{\mathbb{R}^3} e^{ixp}\hat{\psi}(p)\,dp \right|$$

$$\leq (2\pi)^{-3} \int_{\mathbb{R}^3} |\hat{\psi}(p)|\,dp$$

$$= (2\pi)^{-3} \int_{\mathbb{R}^3} (\varepsilon|p|^4+1)^{-1/2} (\varepsilon|p|^4+1)^{1/2} |\hat{\psi}(p)|\,dp$$

$$\leq (2\pi)^{-3} \underbrace{\left(\int_{\mathbb{R}^3} (\varepsilon|p|^4+1)^{-1}dp\right)^{1/2}}_{=(C\,\varepsilon^{-3/4})^{1/2}} \underbrace{\left(\int (\varepsilon|p|^4+1)|\hat{\psi}(p)|^2dp\right)^{1/2}}_{=(\varepsilon\|-\Delta\psi\|_2^2+\|\psi\|_2^2)^{1/2}}$$

$$\leq C\,\varepsilon^{1/8}\|-\Delta\psi\|_2 + C\,\varepsilon^{-3/8}\|\psi\|_2$$

, so given $V = \underbrace{V_2 + V_\infty}_{\in L_2(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)}$

$\forall \psi \in H^2(\mathbb{R}^3)\ \forall \varepsilon>0$:

$$\|V\psi\|_2 \leq \|V_2\psi\|_2 + \|V_\infty\psi\|_2$$

$$\leq \|V_2\|_2\,\|\psi\|_\infty + \|V_\infty\|_\infty\,\|\psi\|_2$$

$$\leq C\|V_2\|_2\,\varepsilon^{1/8}\|-\Delta\psi\|_2 + (C\|V_2\|_2\,\varepsilon^{-3/8} + \|V_\infty\|_\infty)\|\psi\|_2 \quad \square$$

The hydrogen atom: $-\Delta - \frac{1}{|x|}$ is selfadjoint on the domain $H^2(\mathbb{R}^3)$ and essentially self-adjoint e.g. on $C_c^\infty(\mathbb{R}^3)$;

Indeed,
$$-\frac{1}{|x|} = \underbrace{-\mathbb{1}_{B(0,1)}(x)\frac{1}{|x|}}_{\in L^2(\mathbb{R}^3)} - \underbrace{\mathbb{1}_{\mathbb{R}^3\setminus B(0,1)}(x)\frac{1}{|x|}}_{\in L^\infty(\mathbb{R}^3)}$$
$$\in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3).$$

Also $-\Delta \geq 0$, so $-\Delta - \frac{1}{|x|}$ is also bounded from below $\Rightarrow$ "stability".

Many-body Schrödinger operators: $V_1, \dots, V_N : \mathbb{R}^3 \to \mathbb{R}$ measurable,

consider $-\Delta + \sum_{k=1}^{N} V_k$:

$$\left(-\Delta + \sum_{k=1}^{N} V_k\right) \psi(\underset{\in \mathbb{R}^3}{x_1}, \dots, \underset{\in \mathbb{R}^3}{x_N}) = -\Delta\psi(x_1, \dots, x_N) + \sum_{k=1}^{N} V_k(x_k)\psi(x_1, \dots, x_N),$$

$\psi \in L^2(\mathbb{R}^{3N})$ suitable.

$-\Delta + \sum V_k$ operator on $L^2(\mathbb{R}^{3N})$ with domain $H^2(\mathbb{R}^{3N})$.

Kato's Theorem: $V_1, \dots, V_N \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ real valued $\Rightarrow$

$-\Delta + \sum_{k=1}^{N} V_k$ selfadjoint on $H^2(\mathbb{R}^{3N})$, essentially selfadjoint on C_c^∞.

Proof: Show that $\sum_{k=1}^{N} V_k$ (c,d)-bounded wrt $-\Delta$ for some $c<1$, $d\geq 0$.

Let $k \in \{1, \dots, N\}$, $\psi \in H^2(\mathbb{R}^{3N}) \Rightarrow \forall a>0\ \exists b\geq 0$:

$$\|V_k\psi\|_2^2 = \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |V_k(x_k)\psi(x_1, \dots, x_N)|^2 dx_k \right) \underbrace{dx_1 \cdots dx_N}_{\text{no } dx_k}$$

$$\leq 2a^2 \int \cdots \int\int \cdots \int \left(\int_{\mathbb{R}^3} |-\Delta_k \psi|^2 dx_k \right)$$

$$+ 2b^2 \int \cdots \int\int \cdots \int \left(\int_{\mathbb{R}^3} |\psi|^2 dx_k \right)$$

$$= 2a^2 \|-\Delta_k \psi\|_2^2 + 2b^2 \|\psi\|_2^2$$

$$\Rightarrow \|V_k\psi\|_2 \leq \sqrt{2}a \|-\Delta_k\psi\|_2 + \sqrt{2}b\|\psi\|_2$$

$$\Rightarrow \left\|\sum_{k=1}^{N} V_k\psi\right\|_2 \leq \sum_{k=1}^{N} \|V_k\psi\|_2 \leq \sqrt{2}a \sum_k \|-\Delta_k\psi\|_2 + \sqrt{2}b \sum_k \|\psi\|_2$$

$$\leq \sqrt{2}aN \|-\Delta\psi\|_2 + \sqrt{2}bN\|\psi\|_2$$

Choosing $a < \frac{1}{\sqrt{2}N}$, the result follows from Kato-Rellich's thm. $\boxed{}$