<u>Theorem 5.15.</u> For any open subset $\Omega \subseteq \mathbb{R}^k$, the Dirichlet realization A_γ of $-\Delta$ on Ω has a system of positive eigenvalues going to ∞:

$$\lambda_1(A_\gamma) \leq \lambda_2(A_\gamma) \leq \dots \leq \lambda_n(A_\gamma) \leq \dots \longrightarrow \infty,$$

(repeated according to multiplicities),
with associated eigenvectors $\{e_n(A_\gamma)\}_{n\in\mathbb{N}}$ forming an ONB for $L_2(\Omega)$.

- If Ω is <u>contented</u>, the counting function

$$N(t, A_\gamma) := \#\{\text{eigenvalues of } A_\gamma \text{ that are } \leq t\}$$

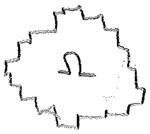
satisfies

(*) $$N(t, A_\gamma) = (2\pi)^{-k}\omega_k \operatorname{vol}(\Omega) t^{k/2} + o(t^{k/2}), \text{ for } t \to \infty.$$

- If Ω is contained in the closure of the union of inner cubes $Q_{j,l}$ for <u>some</u> l, that is,

$$\Omega \subseteq \bigcup_{j=1}^{N_l} Q_{j,l}$$

(so Ω looks like "cubes" on the boundery:



), then

$$N(t, A_\gamma) = (2\pi)^{-k}\omega_k \operatorname{vol}(\Omega) t^{k/2} + \underbrace{O(t^{(k-1)/2})}_{\text{difficult}}, \quad t \to \infty.$$

Note: This last result holds for more general Ω as well, but it is difficult. It was open up until the '70's (general smooth domains) (Hörmander (Fourier integral operators).).

Now let's first recall that the "Dirichlet realization" of $-\Delta$ is the variational operator defined from the sesqui-linear form (as in DifFun 1)

$$s(u,v) = \int_\Omega \nabla u \nabla v \, dx \quad , \quad u,v \in V$$

with $V = H_0^1(\Omega)$ and $H = L_2(\Omega)$,

" $-\Delta$ with boundary condition $u=0$ ".

Actually we know that $D(A_\gamma) = H^2(\Omega) \cap H_0^1(\Omega)$.

Definition: Contented

Ω bounded open set. For each $l \in \mathbb{N}$:

$\{Q_{j,l}\}_{j=1}^{N_l}$ set of the N_l disjoint open cubes, sidelength 2^{-l} and corners at grid-points $2^{-l}(n_1, \dots, n_k)$ that are contained wholly in Ω, i.e.,

$$Q_{j,l} \subseteq \Omega \quad \text{for all } j \leq N_l.$$

We call these cubes the inner cubes and denote

$$W_{in,l} = \bigcup_{j=1}^{N_l} Q_{j,l}.$$

To these cubes we now add exactly so many cubes that we get a set of disjoint cubes containing Ω.

Let N_l' be the smallest number of cubes required so that

$$\Omega \subset \bigcup_{j=1}^{N_l'} \overline{Q}_{j,l}.$$

We call the collection $\{Q_{j,l}\}_{j=1}^{N_l'}$ the outer cubes and denote $W_{out,l} = \left(\bigcup_{j=1}^{N_l'} \overline{Q}_{j,l}\right)^\circ$.

We say that Ω is contented when

$$\limsup_{l\to\infty} \mathrm{vol}(W_{in,l}) = \liminf_{l\to\infty} \mathrm{vol}(W_{out,l}) = \mathrm{vol}(\Omega).$$

Our goal is to prove (*). First we consider the situation where Ω is a cube, say, $Q =]0,\pi[^k$

Here one finds that the functions

$$f_n(x) = \left(\tfrac{2}{\pi}\right)^{k/2} \sin n_1 x_1 \cdot \ldots \cdot \sin n_k x_k \in \mathcal{D}(A_\gamma), \quad n \in \mathbb{N}^k$$

are eigenfunctions for A_γ.

and since the system of functions $\{f_n\}_{n\in\mathbb{N}^k}$ forms an ONB for $L_2(Q)$ this must be all eigenfunctions.

Thus the corresponding eigenvalues

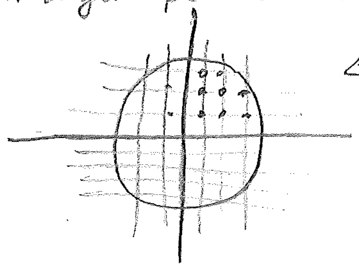
$$\|n\|^2 = n_1^2 + \cdots + n_k^2, \quad n \in \mathbb{N}^k$$

must be all eigenvalues.

All of these will have finite multiplicity.

Now we wish to count the eigenvalues $\leq t$, that means:

"How many positive integer points are there in a ball of radius $\sqrt{t}$?"



we only look at 1st quadrant.

why ?

In this situation the counting function can be shown to satisfy

$$N(t) = 2^{-k}\omega_k t^{k/2} + O(t^{(k-1)/2}), \quad t \to \infty.$$

Recall from lectures (shown in greater generality):

Theorem 5.10. (max-min principle).
When S is selfadjoint ≥ 1 in H with S^{-1} compact, and s is the associated sesquilinear form, with domain V, the eigenvalues of S are described by

$$\lambda_n = \cancel{\min_{\substack{v\in V\setminus\{0\}\\ v\perp e_1,\dots,e_{n-1}}} \frac{s(v,v)}{\|v\|_H^2}} = \max_{\substack{X\subset H\\ \dim X\leq n-1}} \min_{\substack{v\in V\setminus\{0\}\\ v\perp X}} \frac{s(v,v)}{\|v\|_H^2}$$

The formula extends to the case where S is just selfadjoint lower bounded, associated with a coersive symmetric sesquilinear form $s(u,v)$ on a space $V\subset H$ as long as the injection of V into H is compact.

As a (direct) consequence of this we get

Theorem 5.11. Let (H_1, V_1, s_1) and (H_2, V_2, s_2) be triples giving rise to selfadjoint variational operators S_1 resp. S_2.
Assume that
- $V_1 \subset V_2$ with continuous injection
- H_1 is a closed subspace of H_2
- the injections V_i into H_i $(i=1,2)$ are compact
- $s_1(v,v) \geq s_2(v,v)$ for $v\in V_1$.

Then the eigenvalues of S_1 and S_2 satisfy
$$\lambda_n(S_1) \geq \lambda_n(S_2), \quad \text{for all } n\in\mathbb{N}.$$
Equivalently, the countingfunction satisfy
$$N(t;S_1) \leq N(t;S_2), \text{ for all } t\geq 0.$$

"bigger forms gives rise to bigger eigenvalues."

Our goal is now to use 5.11 to investigate what happens to the counting function on a domain which we approximate by a disjoint union of smaller sets.

Let Ω be an bounded open set and let $\{\Omega_j\}_{j\le N}$ be a family of mutually disjoint open sets s.t. $\bigcup_{j\le N}\Omega_j \subset \Omega$.
We identify

$$\bigoplus_{j\le N} L_2(\Omega_j) \simeq L_2(\textstyle\bigcup_{j\le N}\Omega_j) \quad \overset{\text{closed}}{\subseteq} L_2(\Omega)$$

$$\bigoplus_{j\le N} H_0^m(\Omega_j) \simeq H_0^m(\textstyle\bigcup_{j\le N}\Omega_j).$$

(For $-\Delta$ we only need $m=1$, but since it holds for $m>0$ let's just do it more general).

If we then put

$$H_1 = L_2(\cup\Omega_j), \quad V_1 = H_0^m(\cup\Omega_j)$$
$$H_2 = L_2(\Omega), \quad V_2 = H_0^m(\Omega)$$

and let "$s_1 = s_2 = s$" (think about $s(u,v) = (\nabla u, \nabla v)_{L^2}$) we see that all the conditions in 5.11 are satisfied.
Indeed, the injections

$$H_0^m(\Omega) \hookrightarrow L_2(\Omega)$$
$$H_0^m(\Omega_j) \hookrightarrow L_2(\Omega_j)$$
$$H_0^m(\cup\Omega_j) \hookrightarrow L_2(\cup\Omega_j)$$

are compact when $m>0$.

Thus 5.11 tells us that,

$$\left(\sum_{j\le N} N(t, A_{\gamma,\Omega_j}) =\right) N(t, A_{\gamma,\cup\Omega_j}) \le N(t, A_{\gamma,\Omega}),$$

$N(A_1 \oplus A_2) = N(A_1) + N(A_2)$

"eigenfunctions on small domains $\to$ extend by zero to big domain"

i.e.,
(I): the sum of the counting functions for the Dirichlet problems on the Ω_j's is <u>lower</u> than the counting function for the Dirichlet problem on Ω.

Notation: $N(\Omega) := N(t, A_{\gamma,\Omega})$

From this we see that if $\Omega \subset \tilde{\Omega}$ we get
(Skriv: $\tilde{\Omega} \supseteq \Omega \cup (\tilde{\Omega} \setminus \Omega)^{\circ}$)

$$N(\tilde{\Omega}) \geq N(\Omega) + N((\tilde{\Omega} \setminus \Omega)^{\circ}) \geq N(\Omega)$$

and thus

(II) For the Dirichlet problem, an enlargement of the domain increases the counting function.

Note, if we consider the Neumann realization instead of the Dirichlet we can make similar observations using 5.11, and this gives (compare to (I)):

(III) The sum of the counting functions for the Neumann problem on the Ω_j's is higher than the counting function for the Neumann problem on Ω'.

(Here $\Omega' = (\bigcup \overline{\Omega_j})^{\circ}$, and Ω_j's are open cubes (as in the formulation of Theorem 5.15.).

Moreover, we see from 5.11 that

(IV) Replacing a Dirichlet condition by a Neumann condition increases the counting function.

$\longrightarrow$ This follows from 5.11 since $\underset{H_1}{H_0^1(\Omega)} \subseteq \underset{H_2}{H^1(\Omega)}$

Now we are ready to prove 5.15.
The tactic is to "squeeze" the counting function on Ω in between the value of the counting function on the union of inner cubes resp. outer cubes

$$\sum_{j=1}^{N_\ell'} N_\nu(\Omega_j') \geq N_\gamma\Big(\bigcup_{j=1}^{N_\ell'} \Omega_j'\Big) \geq \underbrace{N_\gamma(\Omega)}_{(*)} \geq N_\gamma\Big(\bigcup_{j=1}^{N_\ell} \Omega_j\Big) \geq \sum_{j=1}^{N_\ell} N_\gamma(\Omega_j) \quad (\square)$$

and then use that Ω is contented, so that the limits of left- and right-hand side agree.

To see $(\square)$, note that

$$\sum_{j=1}^{N_\ell'} N_\nu(\Omega_j') \underset{(III)}{\geq} N_\nu\Big(\bigcup_{j=1}^{N_\ell'} \Omega_j'\Big) \underset{(IV)}{\geq} N_\gamma\Big(\bigcup_{j=1}^{N_\ell'} \Omega_j'\Big) \underset{(II)}{\geq} N_\gamma(\Omega)$$

and $$N_\gamma(\Omega) \underset{(II)}{\geq} N_\gamma\Big(\bigcup_{j=1}^{N_\ell} \Omega_j\Big) = \sum_{j=1}^{N_\ell} N_\gamma(\Omega_j) .$$

Both $N_\gamma(\Omega_j)$ and $N_\nu(\Omega_j')$ are known from the first part of the talk, so the left and the right hand side of $(\square)$ can be computed.

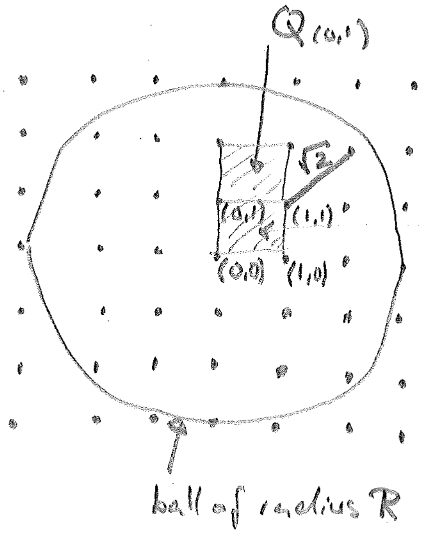
For contented domains, letting $\ell \to \infty$ (i.e. shrinking the cubes to 0) both the left and the right hand side converge to $(*)$, hence $N(t, A_\gamma) = (*)$.

Gauss circle problem

Problem: Count number of integer points in a k-dimensional ball of radius R for $R \to \infty$, i.e. determine

$$N(R) := \left| \{ x \in \mathbb{Z}^k : \|x\| \leq R \} \right| \quad \text{as } R \to \infty.$$

dimension 2:



To $(i,j) \in \mathbb{Z}^2$ associate the square $Q_{(i,j)} = (i,j) + [0,1)^2$.

$Q_{(0,0)}$

$$\Longrightarrow \quad \boxed{B_{R-\sqrt{2}} \subseteq \bigcup_{\|(i,j)\| \leq R} Q_{(i,j)} \subseteq B_{R+\sqrt{2}}}$$

$$\Longrightarrow \quad \underbrace{\mathrm{vol}(B_{R-\sqrt{2}})}_{= \pi R^2 - O(R)} \leq \sum_{\|(i,j)\| \leq R} \mathrm{vol}(Q_{(i,j)}) = N(R) \leq \underbrace{\mathrm{vol}(B_{R+\sqrt{2}})}_{= \pi R^2 + O(R)}$$

$$\Longrightarrow \left| N(R) - \pi R^2 \right| \leq C R \qquad \forall R \geq 1$$

[**Conjecture:** $\left| N(R) - \pi R^2 \right| \leq C_\varepsilon R^{1/2 + \varepsilon} \quad \forall \varepsilon > 0$

Huxley, 2000: True for $\forall \varepsilon \geq 0.1298\ldots$
Hardy, Landau 1910s: False for $\varepsilon = 0$.]

$N_+(R) := \left| \{ x \in \mathbb{N}^2 : \|x\| \leq R \} \right|$ integer points in first quadrant.

$$4 N_+(R) + \underbrace{4 \lfloor R \rfloor + 1}_{\text{integer points on axes}} = N(R) \quad \Longrightarrow \quad N_+(R) = \tfrac{1}{4} N(R) + O(R)$$

dimension k: analogous proof: $\left| N(R) - \mathrm{vol}(B_1) R^k \right| \leq C R^{k-1}$

$$N_+(R) = \left(\tfrac{1}{2}\right)^k N(R) + O(R^{k-1}).$$