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Spectrum of the Laplace operator on S^{n-1}

(M, g) Riemannian manifold

g smooth map $M \ni p \mapsto g_p$ innerpr. on T_pM

κ chart for M

∂_i vectorfield given by $\frac{\partial}{\partial x^i} f \circ \kappa^{-1}(x)$

basis for $T_{\kappa^{-1}(x)}M$. $(f \in C^\infty(M))$

For $f \in C^\infty(M)$, define $df \in \Gamma(T^*(M))$

by $df(v) = v(f)$

Define ∇f by $g(\nabla f, \cdot) = df$ (gradient.)

Laplace operator: Δ diff op M defined by

$$\int_M (\Delta\varphi)\psi \; \omega = -\int_M g(\nabla\varphi, \nabla\psi)\, \omega \quad (\varphi, \psi \in C_c^\infty(M))$$

Riemannian metric

Formula in coordinates:

$$\Delta\varphi = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_j \left(\sqrt{g}\, g^{ij}\, \partial_i \varphi \right),$$

where ∂_i as before

$$\sqrt{g} = \det(\mathrm{mat}\, g)$$

$$g^{ij} = (\mathrm{mat}\, g)^{-1}_{ij}$$

Let $S = S^{n-1}$ isometrically embedded in $\mathbb{R}^n$ as usual.

$$x^i = r\omega^i$$

$$dx^i = \omega^i dr + r\, d\omega^i$$

$$g = \sum_i dx^i \otimes dx^i$$

$$= \sum_i \left((\omega^i)^2 dr \otimes dr + r\omega^i dr \otimes d\omega^i + r\omega^i d\omega^i \otimes dr + r^2 d\omega^i \otimes d\omega^i \right)$$

$$= dr \otimes dr + r\, dr \otimes d\Big(\sum_i \omega^i d\omega^i\Big) + r\, d\Big(\sum_i \omega^i d\omega^i\Big) \otimes dr + r^2 \sum_i d\omega^i \otimes d\omega^i$$

$$= dr \otimes dr + r^2 g_S$$

$$\operatorname{mat} g = \left(\begin{array}{c|c} 1 & \\ \hline & r^2 \operatorname{mat} g_S \end{array}\right), \quad \operatorname{mat} g^{-1} = \left(\begin{array}{c|c} 1 & \\ \hline & \frac{1}{r^2} \operatorname{mat} g_S^{-1} \end{array}\right)$$

$$\sqrt{g} = \sqrt{\det g} = r^{n-1} \sqrt{\det g_S} = r^{n-1} \sqrt{g_S}$$

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^j} \left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^i} \right)$$

$$= r^{1-n} \frac{1}{\sqrt{g_S}} \frac{\partial}{\partial r} \left(r^{n-1} \sqrt{g_S} \frac{\partial f}{\partial r} \right)$$

$$+ r^{1-n} \frac{1}{\sqrt{g_S}} \sum_{i,j=1}^{n-1} \frac{\partial}{\partial y^j} \left(r^{n-3} \sqrt{g_S}\, g_S^{ij} \frac{\partial f}{\partial y^i} \right)$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \Delta_S f$$

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_S$$

__Prop__ Let $\varphi \in C^\infty(S)$ and $\mu \in \mathbb{R}$. The following are equivalent.

(i) $-\Delta_S \varphi = \mu(\mu+n-2)\varphi$

(ii) $\exists\, u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ s.t.

- $u|_S = \varphi$
- $\Delta u = 0$
- u homogeneous of deg μ.

If (i) holds, then u is uniquely determined by φ.

__Proof__: Let $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be homogeneous of deg μ. Then

$$\Delta u(r\omega) = \left(\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_S\right) r^\mu u(\omega)$$

$$= r^{\mu-2}\big(\mu(\mu-1) + \mu(n-1) + \Delta_S\big)u(\omega)$$

$$= r^{\mu-2}\big(\mu(\mu+n-2) + \Delta_S\big)u(\omega)$$

(i)$\Rightarrow$(ii): Take $u(x) = \|x\|^\mu \varphi\left(\frac{x}{\|x\|}\right)$, (ii)$\Rightarrow$(i) immedia

N.B. Every eigenvalue of $-\Delta_S$ are non-negative.

If λ is an e.v., then

$\exists!\ \mu \geq 0$ s.t. $\lambda = \mu(\mu+n-2)$ ⇝ Shubin

and

$\exists!\ \mu \leq -(n-2)$ s.t. $\lambda = \mu(\mu+n-2)$ ⇝ Hörmander.

Let $\mu \leq 2-n$. $\boxed{\text{Want to find all harmonic functions homogeneous of degree } \mu.}$

Lemma: Let $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ be hom. of deg μ.

Then $\exists\, v \in \mathcal{D}'(\mathbb{R}^n)$ s.t. $v\big|_{\mathbb{R}^n \setminus \{0\}} = u$.

If $\mu \notin -\mathbb{N}$, then $\exists!\ v \in \mathcal{D}'(\mathbb{R}^n)$

s.t. $v\big|_{\mathbb{R}^n \setminus \{0\}} = u$ and v hom. of deg μ.

<u>Sketch of proof</u>: Construct $P: C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R}^n \setminus \{0\})$ continuous, linear s.t.

See Hörmander Thm 3.2.3

(i) If $0 \notin \operatorname{supp} \varphi$, then $u(P\varphi) = u(\varphi)$

(ii) $P(t^n t^* \varphi) = t^{-\mu} P\varphi$ if $\mu \notin -\mathbb{N}$.

<u>N.B.</u>: $\nexists\, v \in \mathcal{D}'(\mathbb{R})$ homogeneous of deg $-k$ s.t.

$$v\big|_{\mathbb{R}\setminus\{0\}} = x_+^{-k} = \left(x \mapsto \begin{cases} 0 & x<0 \\ x^{-k} & x>0 \end{cases}\right)$$

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be such that

(i) $\operatorname{supp}(\Delta u) \subseteq \{0\}$

(ii) $u\big|_{\mathbb{R}^n\setminus\{0\}}$ hom. of deg μ.

<u>Lemma</u>: If $v \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp} v \subseteq \{0\}$, then $\exists\, k \in \mathbb{N}_0$ s.t. $v = \sum_{|\alpha| \le k} c_\alpha \partial^\alpha \delta$.

<u>alternatively</u>: $\exists\, P$ polynomial : $v = P(\frac{1}{i}\partial)\delta$

$\Rightarrow \Delta u = P_0(\frac{1}{i}\partial)\delta$ for some pol. P_0

Fundamental Solutions

$\mathcal{D}$ diff. operator on $\mathbb{R}^n$ with constant coeff.

Thm (Ehrenpreis 1954-1955, Malgrange, 1956):

$\exists\, E \in \mathcal{D}'(\mathbb{R}^n)$ s.t. $\mathcal{D}E = \delta$

E is called a fundamental solution

N.B. If $f \in \mathcal{E}'(\mathbb{R}^n)$, then $v = E * f$ is a solution of $\mathcal{D}v = f$.

Fundamental solution for Δ

Want to solve $\Delta E = \delta$.

$\Delta E = \delta \iff -\|\cdot\|^2 \mathcal{F}E = \mathbb{1}$ (assuming $E \in \mathcal{S}'(\mathbb{R}^n)$) $\Leftarrow \mathcal{F}E = \frac{-1}{\|\cdot\|^2}$ (loc. int. for $n \geq 3$) $\iff E = \mathrm{const}\, \|\cdot\|^{2-n}$

E is indeed tempered, since it is homogeneous

Want to solve $\Delta u = P_0(\frac{1}{i}\partial)\delta$

Solutions are $u = P_0(\frac{1}{i}\partial)\delta * E + v$
$= P_0(\frac{1}{i}\partial)E + v$,

where $v \in \mathcal{S}'(\mathbb{R}^n)$ is harmonic

$$\mathcal{F}u = \frac{-P_0}{\|\cdot\|^2} + \mathcal{F}v$$
$$= \frac{-P_0}{\|\cdot\|^2} \quad \text{outside origin}$$

$\Rightarrow \mathcal{F}v$ supported in $\{0\}$.

$\Rightarrow \mathcal{F}v = Q(i\partial)\delta$ for some pol. Q

$\Rightarrow v = Q$

outside 0 : u is hom of deg $\mu \leq 2-n < 0$ (for $n \geq 3$)

E is hom of deg $2-n < 0$.

$\partial^\alpha E$ is hom of deg $2-n-|\alpha|$

$(x \mapsto x^\alpha)$ hom of deg $|\alpha|$

~~mod pol. Q s.t. $Q(\frac{1}{i}\partial)E = 0$ outside $\{0\}$.~~

$\Rightarrow v = Q = 0$, P_0 hom of deg ~~$2-n-\mu$~~ of $u|_{\mathbb{R}^n\setminus\{0\}} = 0$.

Conclusion:

$$u = P(\tfrac{1}{i}\partial)E + v,$$

$$\begin{cases} P \text{ hom. pol. of deg } 2-n-\mu \\ \operatorname{supp} v \subseteq \{0\}. \end{cases}$$

N.B. : $2-n-\mu \in \mathbb{N}_0$ or equivalently $\mu \in 2-n-\mathbb{N}_0$

We have shown (at least the non-triv. impl.)

Prop: Let $u \in \mathcal{D}'(\mathbb{R}^n)$, then the following are equivalent.

(i) $\operatorname{supp}(\Delta u) \subseteq \{0\}$ and $u/_{\mathbb{R}^n\setminus\{0\}}$ hom. of deg μ.

(ii) $\exists$ pol. P, hom. of deg $2-n-\mu$ s.t.

$$u = P(\tfrac{1}{i}\partial)E \text{ outside origin}$$

N.B. 1) In this case $\mu \in 2-n-\mathbb{N}_0$, or $u=0$

2) u is smooth outside

3) $u/_{\mathbb{R}^n\setminus\{0\}} = 0 \iff \mathcal{F}u = \dfrac{-P}{\|\cdot\|^2} + \mathcal{F}v$ is a polynomial

$\iff \exists\, Q$ pol. s.t. $P = \|\cdot\|^2 Q$.

Counting:

Let $\mathcal{H}_{n,k}$ be the space of hom. pol. of deg $k \in \mathbb{N}_0$ in n-variables.

We want to determine $\dim\left(\mathcal{H}_{n,k} / \|\cdot\|^2 \mathcal{H}_{n,k-2}\right)$

$$\dim\left(\mathcal{H}_{n,k}\right) = \frac{(k+n-1)!}{k!\,(n-1)!} = \binom{k+n-1}{k} \quad k \geq 0.$$

$$\begin{cases} \dim\left(\mathcal{H}_{n,k} / \|\cdot\|^2 \mathcal{H}_{n,k-2}\right) = \dim\left(\mathcal{H}_{n,k}\right) - \dim\left(\mathcal{H}_{n,k-2}\right) \\ \qquad = \binom{k+n-1}{k} - \binom{k+n-3}{k-2} \quad (k \geq 2) \\ \dim\left(\mathcal{H}_{n,k} / \|\cdot\|^2 \mathcal{H}_{n,k-2}\right) = \dim\left(\mathcal{H}_{n,k}\right) = \binom{k+n-1}{k} \quad (k = 0, 1) \end{cases}$$

Thm.: $\sigma(-\Delta_S) = \{\lambda_k : \lambda_k = k(k+n-2),\ k \in \mathbb{N}_0\}$

Corresponding eigenfunctions are restrictions to S of $u = P(\frac{1}{i}\partial)E$, where $P \in \mathcal{H}_{n,k}$, $P \notin \|\cdot\|^2 \mathcal{H}_{n,k-2}$

Multiplicity of λ_k is $\binom{k+n-1}{k} - \binom{k+n-3}{k-2}$ $(k \geq 2)$

Let $N(\lambda) = \#\{\lambda_0 \in \sigma(-\Delta_S) : \lambda_0 \leq \lambda\}$ counted with multiplicity.

$$\begin{cases} N(\lambda_k + 0) = \binom{k+n-1}{k} + \binom{k+n-2}{k-1} \\ N(\lambda_k - 0) = \binom{k+n-2}{k-1} + \binom{k+n-3}{k-2} \end{cases}$$

$$N(\lambda_k + 0) - N(\lambda_k - 0) = \binom{k+n-1}{k} - \binom{k+n-3}{k-2}$$

$$= \frac{2k^{n-2}}{(n-2)!}\left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right)$$

$$= \frac{2\lambda_k^{\frac{n-2}{2}}}{(n-2)!}\left(1 + \mathcal{O}\left(\frac{1}{\lambda^{\frac{1}{2}}}\right)\right)$$

This proves that there is no asymptotic formula for $N(\lambda)$ with continuous main term and error $o(\lambda^{\frac{n-2}{2}})$