Isak: Resolvents of pseudodifferential operators

"Shubin": $S^m_d(X, \mathbb{R}^n, \Lambda)$, X compact manifold, $n \in \mathbb{N}$, $\Lambda \subseteq \mathbb{C}$

$a \in S^m_d(X, \mathbb{R}^n, \Lambda) \iff$ • $a(x,\xi,\lambda_0) \in C^\infty(X, \mathbb{R}^n)$ $\forall \lambda_0 \in \Lambda$

• $|\partial_x^\beta \partial_\xi^\alpha a(x,\xi,\lambda_0)| \leq C_{\alpha\beta} (1+|\xi|+|\lambda|^{1/d})^{m-|\alpha|}$

Theorem: $A \in DO^m(X)$, A elliptic wrt. a closed angle Λ (i.e. the principal symbol $a_m(x,\xi)$ does not take values in Λ)

$\Longrightarrow$ a) $\exists$ $R>0$ st $A-\lambda$ invertible for $\lambda \in \Lambda_R = \Lambda \cap \{|\lambda| \geq R\}$ and $(A-\lambda)^{-1} = op$ (classical symbol $\in S^{-m}_m(X, \Lambda_R)$)

b) $\|(A-\lambda)^{-1}\|_{L^2 \to L^2} \leq \frac{C}{|\lambda|}$, $\lambda \in \Lambda_R$.

Example: $p(x,\xi,\lambda) = \langle\xi\rangle + \lambda$

$\partial_\xi^2 (\langle\xi\rangle + \lambda) = \partial_\xi^2 \langle\xi\rangle$ does not depend (in particular: not decay) in λ!

$\partial_\xi^2 (\langle\xi\rangle+\lambda)^{-1}$ only get decay in $\lambda \approx |\xi,\lambda|^{-2}$.

Definition: $n, \nu \in \mathbb{N}$, $m, d \in \mathbb{R}$, Γ sector in $\mathbb{C}$ with vertex 0.

Then define $S^{m,0}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$ as functions $p(x,\xi,\mu) \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^n \times \Gamma)$, holomorphic in $\mathring{\Gamma}$ for $|\xi,\mu| \geq \varepsilon$ and $\partial_z^j p(\cdot,\cdot,\frac{1}{z}) \in S^{m+j}(\mathbb{R}^\nu, \mathbb{R}^n)$ $\forall j \in \mathbb{N}_0$ uniformly in z for $|z| \leq 1$, $\frac{1}{z} \in$ closed subsectors of Γ.

Further define $S^{m,d}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma) = \mu^d S^{m,0}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$, i.e.

$$\partial_z^j \left(z^d p(\cdot,\cdot,\tfrac{1}{z})\right) \in S^{m+j}.$$

Define: $p \in S^{\infty,d}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$ is said to be <u>weakly polyhomogeneous</u> (wphg) if $\exists$ sequence $p_j \in S^{m_j - d, d}$, $m_j \downarrow -\infty$, s.t. p_j is homogeneous in (ξ,μ) for $|\xi| \geq 1$ such that

$$p - \sum_{j=0}^{J-1} p_j \in S^{m_J - d, d}.$$

We always have another expansion:

<u>Theorem:</u> For $p \in S^{m,d}(\mathbb{R}^n, \mathbb{R}^n, \Gamma)$, put $p_{(d,k)}(x,\xi) = \frac{1}{k!}\, \partial_z^k \big|_{z=0} \left(z^d\, p(x,\xi,\tfrac{1}{z}) \right)$

Then $p_{(d,k)} \in S^{m+k}(\mathbb{R}^n, \mathbb{R}^n)$ and $\forall N \in \mathbb{N}$

$$p(x,\xi,\mu) - \sum_{k=0}^{N-1} \mu^{d-k}\, p_{(d,k)}(x,\xi) \in S^{m+N,d-N}(\mathbb{R}^n, \mathbb{R}^n, \Gamma)$$

<u>Example:</u> $\dfrac{1}{\xi^2+\mu^2} = \dfrac{\mu^2}{\xi^2/\mu^2+1} = \mu^2\left(1 - \dfrac{\xi^2}{\mu^2} + \dfrac{\xi^4}{\mu^4} - \dfrac{\xi^6}{\mu^6} + \dots \right)$

Higher terms are of higher and higher order in ξ, but of lower order in μ!

By developing the calculus of these symbol classes, Grubb + Seeley prove that if A is classically polyhomogeneous and elliptic of order m $\Rightarrow \exists\, p \in S^{-m,0} \cap S^{0,-m}$ weakly polyhomogeneous s.t.

$$(a+\mu^m) \circ p - 1 \sim 0 \quad \text{in } S^{-m,0} \cap S^{0,-m}.$$

<u>Theorem:</u> Let $p \sim \sum_{j \in \mathbb{N}_0} p_j$ in $S^{\infty,d}_{wphg}(\mathbb{R}^n, \mathbb{R}^n, \Gamma)$ with degrees $\{m_j \downarrow -\infty\}$. Assume that the p_j with $m_j - d \geq -n$ are also in $S^{m_j',d'}$ for some $m_j' < -n$. Then $op(p)$ has a kernel $K_p(x,y,\mu)$ with an expansion on the diagonal

$$K(x,x,\mu) \overset{|\mu| \to \infty}{\sim} \sum_{j=0}^{\infty} c_j(x)\, \mu^{m_j+n} + \sum_{k=0}^{\infty} \left[c_k'(x) \log(\mu) + c_k''(x) \right] \mu^{d-k}$$

uniformly in closed subsectors. c_j and c_k' are "local" in the sense that they can be computed from the p_j and $c_k''(x)$ is "global": It requires knowledge of the full symbol p.

<u>Proof:</u> By considering $p' = \mu^{-d} p$ we reduce to the case $d=0$. The assumptions assure that the p_j and the remainder $r_J = p - \sum_{j=0}^{J-1} p_j$ are all integrable in ξ, hence the corresponding operators have continuous kernels

$$K_{p_j}(x,y,\mu) = \int_{\mathbb{R}^n} e^{i(x-y)\xi}\, p_j(x,\xi,\mu)\, d\xi$$

$$K_{r_j}(x,y,\mu) = \int_{\mathbb{R}^n} e^{i(x-y)\xi}\, r_j(x,\xi,\mu)\, d\xi$$

Expand r_j in μ: $r_j(x,\xi,\mu) = \sum_{0\le\nu<N} s_\nu(x,\xi)\mu^{-\nu} + O(\langle\xi\rangle^{m_j+N}\mu^{-N})$

Choose $N \in \mathbb{N}$. Find j so large that $m_j+N<-n$; Integrate in ξ to find $K_{r_j}(x,x,\mu) = \sum_{0\le\nu<N} c_{\nu j}(x)\mu^{-\nu} + O(\mu^{-N})$.

For the p_j write $K_{p_j}(x,x,\mu) = \int_{\mathbb{R}^n} p_j(x,\xi,\mu)\, d\xi$

$$= \Big(\underbrace{\int_{|\xi|\ge\mu}}_{1} + \underbrace{\int_{|\xi|\le 1}}_{2} + \underbrace{\int_{1\le|\xi|\le\mu}}_{3}\Big)\, p_j\, d\xi \qquad (|\mu|\ge 1)$$

<u>Concerning 1</u>: p_j is homogeneous $\Rightarrow$ Polar coordinates!

$$\int_{|\xi|\ge|\mu|} p_j\, d\xi = \mu^{m_j+n} \underbrace{\int_{|\xi|\le 1} \left(\frac{|\mu|}{\mu}\right)^{m_j+n} p_j\left(x,\xi,\frac{\mu}{|\mu|}\right) d\xi}_{\substack{\text{independent of } \mu \text{ because} \\ p_j \text{ holomorphic in last argument.}}}$$

<u>2</u>: Write $p_j = \sum_{0\le\nu<M} q_\nu(x,\xi)\mu^{-\nu} + R_M$, $R_M \in S^{m_j+M,-M}$

$$\int_{|\xi|\le 1} p_j(x,\xi,\mu)\, d\xi = \sum_{0\le\nu<M} \mu^{-\nu} \int_{|\xi|\le 1} q_\nu(x,\xi)\, d\xi + O(\mu^{-M})$$

<u>3</u>: Integrating in polar coordinates $\mu^{-\nu}\int_{1\le|\xi|\le|\mu|} q_\nu\, d\xi = \mu^{-\nu} c_\nu(x) \cdot \int_1^{|\mu|} r^{n-1+m_j+\nu}\, dr$

$$= \begin{cases} \mu^{-\nu} c_\nu(x)\left(|\mu|^{m_j+\nu+n} - 1\right), & m_j+\nu+n \ne 0 \\ \mu^{-\nu} c_\nu(x) \log(|\mu|) & \text{otherwise.} \end{cases}$$

Note that $R_M(x,\xi,\mu) = O(|\xi|^{m_j+M}\mu^{-M})$ for $|\xi| \geq 1$.

This holds also for small $|\xi|$ if we extend R_M by homogeneity call this extension R_M^h. Choose M so large that $M+m_j > -n$.

Then
$$\int_{1\leq|\xi|\leq|\mu|} R_M \, d\xi = \underbrace{\int_{|\xi|\leq|\mu|} R_M^h \, d\xi}_{= c(x)\mu^{m_j+M}} - \underbrace{\int_{|\xi|\leq 1} R_M^h \, d\xi}_{= O(\mu^{-M})} \qquad \boxed{\xi}$$

<u>Summary</u>: Under the relevant assumptions

$$\mathrm{Tr}\ (A-\lambda)^{-1} \sim \sum_j c_j \lambda^{m_j} + \sum_{k=0}^{\infty} \left(c_j'' + c_j' \log(\lambda)\right) \lambda^k$$

Using then the transition formulas from Appendix 3 of the lecture, we obtain meromorphic continuation of the zeta-functions associated to A with similar pole structure as in the case of differential operators. Ikehara's theorem extends Weyl's law to pseudodifferential operators.