Lecture 21th dec 2011 WAVE

The Cauchy Problem for the Wave equation

We want to find and analyze the solutions of the Cauchy problem for the wave equation in $\mathbb{R}^n$

$$(\partial_t^2 - \Delta_x) u(x,t) = f(x,t), \quad u(x,0) = u_0(x), \quad (\partial_t u)(x,0) = u_1(x)$$

We consider solutions u s.t. for each t $u(t) \in \mathcal{S}'$. We write $u \in C^0(\mathbb{R}; \mathcal{S}')$ if for each $\varphi \in \mathcal{S}$ the map

$\mathbb{R} \ni t \longmapsto \langle u(t), \varphi \rangle$ is continuous, and in the same way we write $u \in C^k(\mathbb{R}; \mathcal{S}')$ if for each $\varphi \in \mathcal{S}$ the formula

$$\langle \partial_t u(t), \varphi \rangle := \frac{d}{dt} (u(t), \varphi)$$

defines a distribution $\partial_t u \in C^{k-1}(\mathbb{R}, \mathcal{S}')$.

The first result we want to prove is the

THEOREM

Let $u_0, u_1 \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in C^k(\mathbb{R}, \mathcal{S}')$. Then The Cauchy problem

$$\boxed{(\partial_t^2 - \Delta) u(t) = f(t) \,, \; u(0) = u_0 \,, \; \partial_t u(0) = u_1}$$

has a unique solution $u \in C^2(\mathbb{R}, \mathcal{S}')$ given by

$$(*) \quad \hat{u}(t) = \cos|\xi|t \, \hat{u}_0 + \frac{\sin|\xi|t}{|\xi|} \hat{u}_1 + \int_0^t \frac{\sin|\xi|(t-s)}{|\xi|} \hat{f}(s) ds$$

In fact $u \in C^{k+2}(\mathbb{R}, \mathcal{S}')$ and

(i) If $u_0, u_1 \in \mathcal{S}$ and $f \in C^k(\mathbb{R}; \mathcal{S})$ then $u \in C^{k+2}(\mathbb{R}, \mathcal{S})$

(ii) If $u_0 \in H^{s+1}$, $u_1 \in H^s$ and $f \in C^0(\mathbb{R}, H^s)$ then $u \in C^0(\mathbb{R}; H^{s+1}) \cap C^1(\mathbb{R}; H^s) \cap C^2(\mathbb{R}, H^{s-1})$

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proof

For a point $t \in \mathbb{R}$ we can take the Fourier transform in the spatial variables to get

$$\left(\frac{d^2}{dt^2} + |\xi|^2\right) \hat{u}(t) = \hat{f}(t), \quad \hat{u}(0) = \hat{u}_0 \,, \; \frac{d}{dt}\hat{u}(0) = \hat{u}_1$$

The homogeneous part of this equation is solved in high shool, and using the Theorem of differentiating under the integral sign one sees that $(*)$ solves the equation

Since the solution is written in terms of particular functions, we collect information on those in a lemma. **WAVE**

Lemma

The functions

$$C(t,\xi) = \cos t|\xi| \qquad S(t,\xi) = \frac{\sin t|\xi|}{|\xi|}$$

are smooth on $\mathbb{R}^{n+1}$ and slowly increasing in t (i.e. all derivatives have at most polynomial growth). Moreover both can be extended to entire functions in the ξ-variable with estimates

$|C(t,\xi)| \le e^{|t|\cdot|\mathrm{Im}\,\xi|}$ and $|S(t,\xi)| \le |t| e^{|t|\cdot|\mathrm{Im}\,\xi|}$

Finally, for $\xi \in \mathbb{R}^n$ we have

$|C(t)| \le 1$ and $\langle\xi\rangle |S(t)| \le (1+t^2)^{1/2}$

proof of lemma

note that

$$C(t,\xi) = \sum_{k=0}^{\infty} \frac{(-t^2|\xi|^2)^k}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-t^2(\xi_1^2+\cdots+\xi_n^2))^k}{(2k)!}$$ which can obviously be extended to all of $\mathbb{C}^n$.

Relatively clear. Only thing not obvious is which follows from Euler's formulas and noting that $S(t) = \int_0^t C(s)\,ds$ and which is seen by squaring:

$$\langle\xi\rangle^2 |S(t)|^2 = \sin^2 t|\xi| + t^2 \frac{\sin^2 t|\xi|}{t^2|\xi|^2} \le 1+t^2$$

We now adress the remainder of the claims of the theorem:

Uniqueness: Suppose $u \in C^2(\mathbb{R}, \mathcal{S}')$ solves the problem. Now consider the distribution

$\mathcal{U}(s,t) = C((t-s)\hat{u}(s) + S(t-s)(\partial_t \hat{u})(s)$. We have

$\hat{u}(t) = \mathcal{U}(t,t) = \mathcal{U}(0,t) + \int_0^t \partial_s \mathcal{U}(s,t)\,ds$,

Using the fundamental theorem of calculus

But we have

$$\partial_s \mathcal{U}(s,t) = \partial_s[C(t-s)]\hat{u}(s) + \cancel{C(t-s)\partial_s \hat{u}(s)} + \cancel{\partial_s[S(t-s)]}\partial_t \hat{u}(s) + S(t-s)(\partial_t^2 \hat{u})(s)$$

$$= S(t-s)\left(|\xi|^2 + \partial_t^2\right)\hat{u}(s) = S(t-s)\hat{f}(s)$$

Which proves the claim.

[I think this can be proven if one knows some uniqueness theorems for 2nd order ODE's]

To prove that $u \in C^{k+2}(\mathbb{R}, \mathcal{S}')$ we compute:

$$\partial_t \hat{u}(t) = -|\xi|^2 S(t)\hat{u}_0 + C(t)\hat{u}_1 + \int_0^t C(t-s)\hat{f}(s)\,ds + 0$$

$$\partial_t^2 \hat{u}(t) = -|\xi|^2 \hat{u}(t) + \hat{f}(t),$$

and since $f \in C^k(\mathbb{R}, \mathcal{S}')$ this proves the claim

In view of these calculations, if $u_0, u_1 \in \mathcal{S}$, $f \in C^k(\mathbb{R}, \mathcal{S})$ then it is clear that $u \in C^k(\mathbb{R}, \mathcal{S})$. For the last claim we calculate

$$\langle\xi\rangle^{s+1}|\hat{u}(t)| \le \langle\xi\rangle^{s+1}|c(t)\hat{u}_0| + \langle\xi\rangle^{s+1}|s(t)\hat{u}_1| + \langle\xi\rangle^{s+1}\left|\int_0^t s(t-r)\hat{f}(r)\,dr\right|$$

$$\le \langle\xi\rangle^{s+1}|\hat{u}_0| + (1+t^2)^{1/2}\langle\xi\rangle^{s}|\hat{u}_1| + \left|\int_0^t \langle\xi\rangle s(t-r)\langle\xi\rangle^{s}\hat{f}(r)\,dr\right|$$

$$\le \underbrace{\langle\xi\rangle^{s+1}|\hat{u}_0|}_{\in L^2} + (1+t)^{1/2}\underbrace{\langle\xi\rangle^{s}|\hat{u}_1|}_{\in L^2} + \int_0^t (1+(t-r)^2)^{1/2}\underbrace{\langle\xi\rangle^{s}|\hat{f}(r)|}_{\in L^2}\,dr$$

The proof for $u \in C^1(\mathbb{R}, H^s) \cap C^2(\mathbb{R}, H^{s-1})$ is similar, just using the formulas for $\partial_t \hat{u}$ and $\partial_t^2 \hat{u}$ just derived.

In the case $f = 0$ similar computations gives us conservation of energy.

COROLLARY If $u_0 \in H^{s+1}$, $u_1 \in H^s$ and $f = 0$, the energy $E_s(t) = \|\partial_t u(t)\|_s^2 + \sum_{j=1}^{n} \|\partial_j u(t)\|_s^2$ is independent of t.

Proof

We calculate:

$$|\partial_t \hat{u}(t)|^2 + \sum_{j=1}^n |\widehat{\partial_j u}(t)|^2 = \left|-|\xi|^2 s(t)\hat{u}_0 + c(t)\hat{u}_1\right|^2 + |\xi|^2 |\hat{u}(t)|^2$$

$$= \left|-|\xi|^2 s(t)\hat{u}_0 + c(t)\hat{u}_1\right|^2 + |\xi|^2 \left|c(t)\hat{u}_0 + s(t)\hat{u}_1\right|^2$$

arithmetic w. complex numbers $\hat{u}$

$$= \left(c^2(t) + |\xi|^2 s^2(t)\right)\left(|\hat{u}_1|^2 + |\xi|^2 |\hat{u}_0|^2\right)$$

$$= |\hat{u}_0|^2 + |\xi|^2 |\hat{u}_1|^2$$

Now integrate!

We now turn to the last result of today namely the phenomenon of finite propagation speed. I.e after a time t the solution knows only of the initial condition finitely far away. More precisely

THEOREM

Let $u_0, u_1 \in \mathcal{S}'$, $f=0$ and $u \in C^\infty(\mathbb{R}, \mathcal{S}')$ the solution to the Cauchy problem. Then

(i) If $\underline{\operatorname{supp} u_0 \cup \operatorname{supp} u_1}$ does not intersect the ball $B(x,|t|) = \{y \in \mathbb{R}^n \mid \|x-y\| \leq |t|\}$ then $(t,x) \notin \operatorname{supp} u$

(ii) If $\operatorname{sing\,supp} u_0 \cup \operatorname{sing\,supp} u_1$ does not intersect the sphere $S(x,|t|) = \{y \in \mathbb{R}^n \mid \|x-y\| = |t|\}$ then $(t,x) \notin \operatorname{sing\,supp} u$

Since the wave operator is invariant under translations and reversal of time we can assume $x=0$, $t \geq 0$.

Proof Let $B_t = \{y \in \mathbb{R}^n \mid |y| \leq t\}$.

(i) If $(\operatorname{supp} u_0 \cup \operatorname{supp} u_1) \cap B_t = \emptyset$ we have $u_0 = u_1 = 0$ on B_T° for some $T > t$. For $0 \leq s < T$ we prove that $u(s) = 0$ on B_{T-s}° $\rightsquigarrow$ this will certainly imply $(x,t) \notin \operatorname{supp} u$.

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We calculate: Assume $\operatorname{supp}\varphi \subseteq B^{o}_{T-s}$ $\varphi \in C_0^\infty(B^{o}_{T-s})$ (since $\hat{u}(t) = C(t)\hat{u}_0 + S(t)\hat{u}_1$)

$$(u(s), \varphi) = (2\pi)^{-n}(\hat{u}(s), \hat{\varphi}) = (2\pi)^{-n}\left[(\hat{u}_0, \overline{C(s)}\hat{\varphi}) + (\hat{u}_1, \overline{S(s)}\hat{\varphi})\right]$$

Actually one has $\operatorname{supp}\varphi \subseteq B_{T-s-\varepsilon}$ for some $\varepsilon > 0$. By the Paley-Wiener theorem $\hat{\varphi}$ can be extended to an entire fct with $|\hat{\varphi}(\xi)| \leq C_N(1+|\xi|^2)^{-N} e^{(T-s-\varepsilon)|\operatorname{Im}\xi|}$

Using the Lemma we see that

$\Phi_0 = C(s)\hat{\varphi}$ $\Phi_1 = S(s)\hat{\varphi}$ can also be extended s.t.

$|\Phi_i(\xi)| \leq C_N(1+|\xi|^2)^{-N} e^{(T-\varepsilon)|\operatorname{Im}\xi|}$. Using Paley-Wiener again we get fcts φ_0, φ_1 with support $\subseteq B_{T-\varepsilon}$ s.t. $\hat{\varphi}_0 = \Phi_0$, $\hat{\varphi}_1 = \Phi_1$, so we get

$$= (2\pi)^{-n}\left[(\hat{u}_0, \Phi_0) + (\hat{u}_1, \Phi_1)\right]$$

$$= \langle u_0, \varphi_0 \rangle + \langle u_1, \varphi_1 \rangle = 0 \text{ since}$$

$u_0 = u_1 = 0$ on B_T. Thus

$u = 0$ on B_{T-s} as desired

(ii) Assume $(\operatorname{sing\,supp} u_0 \cup \operatorname{sing\,supp} u_1) \cap S_t = \emptyset$ WAVE

We first prove that u is smooth near $\{(y,s) \in \mathbb{R}^n \times \mathbb{R} \mid |y| = t, s = 0\}$.

Indeed, if $|y| = t$ then $y \notin \operatorname{sing\,supp} u_0 \cup \operatorname{sing\,supp} u_1$ by assumption ~ hence find $v_0, v_1 \in \mathcal{S}$ s.t. $u_0 = v_0$ and $u_1 = v_1$ in a nbh of y

The Cauchy problem with boundary data v_0, v_1 now has a solution $v \in C^\infty(\mathbb{R}, \mathcal{S})$

From part (i) $(0,y) \notin \operatorname{supp}(u-v)$ since $u-v$ is the solution to the Cauchy problem with data $u_0 - v_0$, $u_1 - v_1$ the combined support of which does not intersect $B(y,0)$. Hence $u = v$ in a nbh of $(y,0)$ which proves $(y,0) \notin \operatorname{sing\,supp} u$.

To prove that indeed $(0,t) \notin \operatorname{sing\,supp} u$ we transport a supposed singularity at $(0,t)$ to $(y,0)$ to obtain a contradiction.

If u were singular at $(t,0)$ there would be a direction $(\xi,\tau) \in \mathbb{R}^{n+1} \setminus \{0\}$ s.t.

$p = ((0,t),(\xi_0,\tau_0)) \in WF\,u$. Since

$$WF\,u \subseteq WF((\partial_t^2 - \Delta)u) \cup \operatorname{Char}(\partial_t^2 - \Delta) \text{ and}$$

$(\partial_t^2 - \Delta)u = 0$ we have that **WAVE**

p is in the characteristic set for the wave operator i.e $\tau = \pm|\xi|$

We now employ the following theorem which can be found e.g in the notes on wave front sets

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THEOREM (The propagation theorem)

If $P \in Op(S^m)$ has real principal symbol p_m, $u \in \mathcal{D}'$ and P properly supported OR $u \in \mathcal{E}'$. Then if I is any connected interval of a bicharacteristic of the function $p_m(x,\xi)$ not intersecting $WF(Pu)$ then either

$$I \subseteq WF(u) \quad \text{OR} \quad I \cap WF(u) = \emptyset$$

Bicharacteristics are given by solutions to the system

$$(*)(*)\quad \dot{\xi} = -\frac{\partial p_m}{\partial x} \qquad \dot{x} = \frac{\partial p_m}{\partial \xi}$$

Since $WF(Pu) = \emptyset$ in this case, and $p \in WF(u)$ we have that the entire bicharacteristic curve passing through p is in $WF(u)$

(**) In this case amounts to WAVE

$$\frac{d}{dr}\big(x(r),t(r)\big) = \big(-2\xi(r), 2\tau(r)\big)$$

$$\frac{d}{dr}\big(\xi(r),\tau(r)\big) = 0 \qquad \Big(\big(x(0),t(0)\big),\big(\xi(0),\tau(0)\big)\Big) = \big((0,t),(\xi_0,\tau_0)\big)$$

which is immedeatelly solved to get

$$\big((x,t),(\xi,\tau)\big) = \big((t-2\tau_0 r, 2\xi_0 r),(\xi_0,\tau_0)\big)$$

Now we are ready, since for $r = t/2\tau_0$ we get

$t(r)=0$ and $|x(r)| = t|\xi_0/\tau_0| = t$!

Hence $(0,x) \in \text{sing supp } u$ with $|x| = |t|$