Verify that if $A^* = A$, then

$$K(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \varphi_j(x) \overline{\varphi_j(y)} \tag{13.35}$$

and the θ-function

$$\theta(t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} = \int_M K(t,x,x)\, dx \tag{13.36}$$

has the asymptotic expansion

$$\theta(t) \sim \sum_{j=0}^{\infty} \alpha_j t^{\frac{j-n}{m}}. \tag{13.37}$$

Express index A in terms of the θ-functions of the operators A^*A and AA^*.

Problem 13.5. Let E be a hermitean vector bundle on a closed manifold M with smooth positive density and let A be an elliptic self-adjoint differential operator mapping $C^\infty(M,E)$ into $C^\infty(M,E)$ (not necessarily semibounded). Consider the function

$$\eta_A(z) = \sum_{\lambda} (\operatorname{sign} \lambda) \cdot |\lambda|^z, \tag{13.38}$$

where the sum runs over all the eigenvalues of A. Show that the series (13.38) converges absolutely for $\operatorname{Re} z < -n/m$ and the function defined by it, $\eta_A(z)$, may be continued to the whole complex z-plane as a meromorphic function with simple poles at $z_j = \frac{j-n}{m}$, $j = 0, 1, 2, \ldots$. Express the residues at these poles via the symbol of A.

Hint. Express $\eta_A(z)$ in terms of $\zeta'_A(z)$ and $\zeta''_A(z)$ where $\zeta'_A(z)$ and $\zeta''_A(z)$ are two ζ-functions of A, obtained by different choices of the branch for λ^z with cuts along the upper and lower semi axes of the imaginary axis.

§14. The Tauberian Theorem of Ikehara

14.1 Formulation. The Tauberian theorem of Ikehara allows us to deduce from the fact that the ζ-function is meromorphic asymptotic formulae for $N(t)$ as $t \to +\infty$ or for λ_k as $k \to +\infty$ (cf. §13). Let us give its exact formulation.

Theorem 14.1. *Let $N(t)$ be a non-decreasing function equal to 0 for $t \leq 1$ and such that the integral*

$$\zeta(z) = \int_1^{\infty} t^z \, dN(t) \tag{14.1}$$

converges for $\operatorname{Re} z < -k_0$, *where* $k_0 > 0$ *and the function*

$$\zeta(z) + \frac{A}{z + k_0}$$

can be extended by continuity to the closed half-plane $\operatorname{Re} z \leqq -k_0$. *We will assume that* $A \neq 0$. *Then, as* $t \to +\infty$ *we have*

$$N(t) \sim \frac{A}{k_0} t^{k_0} \tag{14.2}$$

(*recall that* $f_1(t) \sim f_2(t)$ *as* $t \to +\infty$ *means that* $\lim\limits_{t \to +\infty} f_1(t)/f_2(t) = 1$).

(The convergence of the integral in (14.1) for $\operatorname{Re} z < -k_0$ easily follows from a weaker condition. Namely, it suffices to suppose that the integral converges for $\operatorname{Re} z < -k_1$ for some k_1 and the function $\zeta(z)$ thus expressed can be holomorphically continued to the half-plane $\operatorname{Re} z < -k_0$).

Corollary 14.1. *Suppose that the function* $\zeta(z)$, *defined for* $\operatorname{Re} z < -k_0$ *by* (14.1), *can be meromorphically continued into the larger half-plane* $\operatorname{Re} z < -k_0 + \varepsilon$, *where* $\varepsilon > 0$, *so that on the line* $\operatorname{Re} z = -k_0$ *there is a single and moreover simple pole at* $-k_0$ *with residue* $-A$. *Then the asymptotic formula* (14.2) *holds.*

14.2 Beginning of the proof of Theorem 14.1: The reductions.

1st reduction. It is convenient to consider instead of $\zeta(z)$ the function $f(z) = \zeta(-z)$. We then obtain

$$f(z) = \int_1^\infty t^{-z} \, dN(t), \tag{14.3}$$

where the integral converges for $\operatorname{Re} z > k_0$ and the function $f(z) - \dfrac{A}{z - k_0}$ is continuous for $\operatorname{Re} z \geqq k_0$.

2nd reduction: Reduction to the case $k_0 = 1$. By introducing the function $f_1(z) = f(k_0 z)$, we obtain

$$f_1(z) = \int_1^\infty t^{-k_0 z} \, dN(t) = \int_1^\infty \tau^{-z} \, dN_1(\tau),$$

where $N_1(\tau) = N(\tau^{1/k_0})$. Since

$$f(k_0 z) - \frac{A}{k_0 z - k_0} = f_1(z) - \frac{A}{k_0} \cdot \frac{1}{z - 1}$$

and since $N(t) \sim \dfrac{A}{k_0} t^{k_0}$ is equivalent to $N_1(\tau) \sim \dfrac{A}{k_0} \tau$, then Theorem 14.1 reduces to the following statement:

Let $N(t)$ be a non-decreasing function and let the integral

$$f(z) = \int_1^\infty t^{-z}\, dN(t) \tag{14.4}$$

be convergent for $\operatorname{Re} z > 1$, where $f(z) - \frac{A}{z-1}$ is continuous for $\operatorname{Re} z \geqq 1$. Then

$$N(t) \sim At \quad \text{as} \quad t \to +\infty\,. \tag{14.5}$$

Note that from the continuity of $f(z) - \frac{A}{z-1}$ for $\operatorname{Re} z \geqq 1$ and the fact that $f(z) \geqq 0$ for real $z \geqq 1$, it follows that $A > 0$. Changing $N(t)$ for $A^{-1}N(t)$, which results in changing $f(z)$ for $A^{-1}f(z)$, we see that it suffices to show the statement for $A = 1$.

3rd reduction. Let us pass from the Melin transformation to the Laplace transformation, i.e. make a change of variables $t = e^x$. Put $N(e^x) = \varphi(x)$. We then see that φ (x) is a non-decreasing function, equal to zero for $x < 0$ and that the integral

$$f(z) = \int_0^\infty e^{-zx}\, d\varphi(x) \tag{14.6}$$

converges for $\operatorname{Re} z > 1$ and $f(z) - \frac{1}{z-1}$ is continuous for $\operatorname{Re} z \geqq 1$. We must show that

$$\lim_{x \to +\infty} e^{-x}\varphi(x) = 1\,. \tag{14.7}$$

4th reduction. Denote $H(x) = e^{-x}\varphi(x)$. The $\varphi(x)$ is non-decreasing if and only if

$$H(y) \geqq H(x)\, e^{x-y} \quad \text{for} \quad y \geqq x\,. \tag{14.8}$$

Integrating by parts in (14.6) gives, for $\operatorname{Re} z > 1$

$$f(z) = z\int_0^\infty e^{-zx}\varphi(x)\,dx = z\int_0^\infty e^{-(z-1)x}\, H(x)\,dx \tag{14.9}$$

Now put $z = 1 + \varepsilon + it$, where $\varepsilon > 0$ and t is real. Note that

$$\int_0^\infty e^{-(z-1)x}\,dx = \frac{1}{z-1},$$

therefore, (14.9) implies

$$\frac{f(z)}{z} - \frac{1}{z-1} = \int_0^\infty e^{-(z-1)x}(H(x)-1)\,dx.$$

Since

$$\frac{f(z)}{z} - \frac{1}{z-1} = \frac{1}{z}\left(f(z) - \frac{1}{z-1} - 1\right),$$

then, putting

$$h_\varepsilon(t) = \frac{1}{z}\left(f(z) - \frac{1}{z-1} - 1\right)\bigg|_{z=1+\varepsilon+it}, \tag{14.10}$$

we obtain

$$h_\varepsilon(t) = \int_0^\infty e^{-\varepsilon x - itx}(H(x)-1)\,dx. \tag{14.11}$$

We may now give the following reformulation of Theorem 14.1.

Theorem 14.1′. *Let $H(x)$ be a function, equal to 0 for $x<0$ and satisfying (14.8) for all real x and y. Assume that the integral (14.11) converges absolutely for any $\varepsilon>0$ and the function $h_\varepsilon(t)$ defined by it, is such that the limit*

$$\lim_{\varepsilon\to 0} h_\varepsilon(t) = h(t) \tag{14.12}$$

exists and is uniform on any finite segment $|t| \leqq 2\lambda$. Then

$$\lim_{x\to+\infty} H(x) = 1. \tag{14.13}$$

Remark. If $H(x)$ tends to 1 sufficiently quickly (if e.g. $H(x)-1 \in L^1([0,+\infty))$), then we obtain (14.12) from (14.13) by passing to the limit under the integral sign, which one may do in view of the dominated convergence theorem (the function $h(t)$ then equals the Fourier transform of $\theta(x)(H(x)-1)$, $\theta(x)$ the Heaviside function). In some sense, the Tauberian condition (14.8) allows one to invert this statement.

14.3 The basic lemma. It is clear that in order to prove Theorem 14.1′ we have to somehow express $H(x)-1$ in terms of $h(t)$ which, formally, is possible by the inverse Fourier transformation. However, we know nothing about the behaviour of $h(t)$ as $t\to+\infty$ or about the nature of the convergence of $h_\varepsilon(t)$ to $h(t)$ on the whole line and it is therefore necessary, to begin with, to multiply the limit equality (14.12) with a finite cut-off function $\tilde{\varrho}(t)$. These considerations, linked to the convenience of having transformations with positive kernels (of the Fejér type), demonstrate that it is convenient to consider $\tilde{\varrho}(t)$ to be the Fourier transform of a non-negative function $\varrho(v) \in L^1(\mathbb{R})$:

$$\tilde{\varrho}(t) = \int e^{-itv}\varrho(v)\,dv. \tag{14.14}$$

We shall assume that $\tilde{\varrho}(t)$ is a continuous function with compact support such that $\tilde{\varrho}(0)=1$, $\varrho(v)\geqq 0$ and $\varrho(v)\in L^1(\mathbb{R})$. From this it follows that

$$\int_{-\infty}^{+\infty} \varrho(v)\,dv = 1\,. \tag{14.15}$$

The existence of a function $\tilde{\varrho}(t)$ of the type described may be shown in the same way as in 6.3 (at the beginning of the proof of Theorem 6.3 a function $\tilde{\varrho}(t) \in C_0^\infty(\mathbb{R}^1)$ is constructed which satisfies all these requirements). We can also explicitly define $\tilde{\varrho}(t)$, putting

$$\tilde{\varrho}(t) = \begin{cases} 1 - \dfrac{|t|}{2}\,; & |t| \leqq 2\,, \\ 0 \quad ; & |t| > 2\,. \end{cases}$$

Then indeed, for a fixed $v \neq 0$ we have

$$\varrho(v) = \int_{-2}^{2} e^{itv}\left(1-\frac{|t|}{2}\right)đt = -\int_{-2}^{2}\frac{e^{itv}}{iv}\,đ\left(1-\frac{|t|}{2}\right) + \frac{e^{itv}}{2\pi iv}\left(1-\frac{|t|}{2}\right)\Bigg|_{-2}^{2}$$

$$= \int_{-2}^{2}\frac{e^{itv}}{2iv}\,\mathrm{sign}\,t\,đt = -\frac{e^{itv}}{4\pi v^2}\Bigg|_0^2 - \frac{e^{itv}}{4\pi v^2}\Bigg|_0^{-2} = \frac{1-\cos 2v}{2\pi v^2} = \frac{1}{\pi}\,\frac{\sin^2 v}{v^2}\,,$$

from which all the necessary properties of $\varrho(v)$ are obvious.

Lemma 14.1. *For any fixed* $\lambda > 0$

$$\lim_{y\to+\infty}\int_{-\infty}^{+\infty} H\left(y-\frac{v}{\lambda}\right)\varrho(v)\,dv = 1\,. \tag{14.16}$$

Proof. 1. Put $\tilde{\varrho}_\lambda(t) = \tilde{\varrho}(t/\lambda)$ and $\varrho_\lambda(v) = \lambda\varrho(\lambda v)$ so that $\tilde{\varrho}_\lambda(t)$ is the Fourier transform of $\varrho_\lambda(v)$. It is clear that

$$\int_{-\infty}^{+\infty} H\left(y-\frac{v}{\lambda}\right)\varrho(v)\,dv = \int_{-\infty}^{+\infty} H(y-v)\,\varrho_\lambda(v)\,dv\,, \tag{14.17}$$

and since $\varrho_\lambda(v)$ possesses the same properties as $\varrho(v)$, it suffices to prove (14.16) for $\lambda = 1$.

2. Putting $F_\varepsilon(t) = \tilde{\varrho}(t)\,h_\varepsilon(t)$, we compute the inverse Fourier transform of the function $F_\varepsilon(t)$ with compact support, taking into account that $\tilde{\varrho}(t)$ and $h_\varepsilon(t)$ are the Fourier transforms of the absolutely integrable functions $\varrho(v)$ and $\theta(v)\;(H(v)-1)\,e^{-\varepsilon v}$:

$$\int_{-\infty}^{+\infty} e^{ity}F_\varepsilon(t)\,đt = \int_{-\infty}^{+\infty} e^{ity}\tilde{\varrho}(t)\left[\int_0^\infty (H(x)-1)\,e^{-\varepsilon x - itx}\,dx\right]đt$$

$$= \int_0^{+\infty} (H(x)-1)e^{-\varepsilon x}\left[\int_{-\infty}^{+\infty} \tilde{\varrho}(t)\, e^{it(y-x)}\, đt\right]dx \qquad (14.18)$$

$$= \int_0^{+\infty} (H(x)-1)\, e^{-\varepsilon x}\, \varrho(y-x)\, dx\,.$$

As a result, as one might have anticipated, we obtain a convolution and we have made sure that (14.18) holds everywhere and in the usual sense (the change in the order of integration is permitted by the Fubini's theorem).

Let us now rewrite (14.18) in the form

$$\int_{-\infty}^{+\infty} e^{ity} F_\varepsilon(t)\, đt + \int_0^{\infty} e^{-\varepsilon x}\varrho(y-x)\, dx = \int_0^{\infty} H(x)\, e^{-\varepsilon x}\varrho(y-x)\, dx \qquad (14.19)$$

and take the limit as $\varepsilon \to +0$. Since $\operatorname{supp} F_\varepsilon \subset \operatorname{supp} \tilde{\varrho}$ and $F_\varepsilon(t) \to F(t)$ uniformly in $t \in \operatorname{supp} \tilde{\varrho}$ (here $F(t) = \tilde{\varrho}(t)\, h(t)$), then the first integral on the left-hand side has a limit as $\varepsilon \to +0$ for any y. The same also holds for the second integral (e.g. by the dominated convergence theorem). Therefore, the integral on the right-hand side of (14.19) has for any y a limit as $\varepsilon \to +0$. Since $H(x)e^{-\varepsilon x}\varrho(y-x)$ converges monotonely as $\varepsilon \to +0$ to $H(x)\,\varrho(y-x)$, we get

$$\int_{-\infty}^{+\infty} e^{ity} F(t)\, đt + \int_0^{\infty} \varrho(y-x)\, dx = \int_0^{\infty} H(x)\, \varrho(y-x)\, dx\,. \qquad (14.20)$$

Now let y tend to $+\infty$. By the Riemann lemma

$$\lim_{y\to+\infty} \int_{-\infty}^{+\infty} e^{ity} F(t)\, đt = 0\,.$$

In addition, it is clear that $\lim\limits_{y\to+\infty} \int_0^{\infty} \varrho(y-x)\, dx = 1$. Therefore, it follows from (14.20) that

$$\lim_{y\to+\infty} \int_0^{\infty} H(x)\, \varrho(y-x)\, dx = 1\,. \qquad (14.21)$$

But

$$\int_0^{\infty} H(x)\varrho(y-x)\, dx = \int_{-\infty}^{+\infty} H(x)\, \varrho(y-x)\, dx = \int_{-\infty}^{+\infty} H(y-v)\, \varrho(v)\, dv\,,$$

so that (14.21) implies the statement of the lemma. □

14.4 Proof of Theorem 14.1′. 1. First, we show that

$$\overline{\lim_{y\to+\infty}} H(y) \leqq 1 . \tag{14.22}$$

Since

$$\int_{-a}^{a} H\left(y-\frac{v}{\lambda}\right) \varrho(v)\, dv \leqq \int_{-\infty}^{+\infty} H\left(y-\frac{v}{\lambda}\right) \varrho(v)\, dv$$

for any $a>0$, it follows from Lemma 14.1 that

$$\overline{\lim_{y\to+\infty}} \int_{-a}^{a} H\left(y-\frac{v}{\lambda}\right) \varrho(v)\, dv \leqq 1 . \tag{14.23}$$

Now, in view of the Tauberian condition (14.8) we have

$$H\left(y-\frac{v}{\lambda}\right) \geqq H\left(y-\frac{a}{\lambda}\right) e^{-\frac{2a}{\lambda}} \quad \text{for} \quad v\in[-a,a] .$$

Now, it follows from (14.23) that

$$\overline{\lim_{y\to+\infty}} H\left(y-\frac{a}{\lambda}\right) e^{-\frac{2a}{\lambda}} \int_{-a}^{a} \varrho(v)\, dv \leqq 1 ,$$

or

$$\overline{\lim_{y\to+\infty}} H(y) \leqq e^{\frac{2a}{\lambda}} \left(\int_{-a}^{a} \varrho(v)\, dv\right)^{-1} . \tag{14.24}$$

Inequality (14.24) holds for any $a>0$ and $\lambda>0$. Let $a\to+\infty$ and $\lambda\to+\infty$ in this inequality in such a way that $a/\lambda\to 0$. Then we obtain the required estimate (14.22) from (14.24).

2. We will now verify that

$$\underline{\lim}_{y\to+\infty} H(y) \geqq 1 . \tag{14.25}$$

To begin with, note that (14.22) implies the boundedness of $H(y)$:

$$|H(y)| \leqq M , \qquad y\in\mathbb{R}^1 , \tag{14.26}$$

in view of which

$$\int_{|v|\geqq b} H\left(y-\frac{v}{\lambda}\right) \varrho(v)\, dv \leqq \alpha(b) , \tag{14.27}$$

where $\alpha(b) \to 0$ as $b \to +\infty$ (this means, in particular that the integral on the left-hand side of (14.27) approaches 0 as $b \to +\infty$, uniformly in y and λ).

Since

$$\int_{-\infty}^{+\infty} H\left(y - \frac{v}{\lambda}\right) \varrho(v)\, dv = \int_{-b}^{b} \dots + \int_{|v| \geq b} \dots,$$

then, by (14.27) and Lemma 14.1 we obtain for arbitrary $b > 0$

$$\varliminf_{y \to +\infty} \int_{|v| \leq b} H\left(y - \frac{v}{\lambda}\right) \varrho(v)\, dv \geqq 1 - \alpha(b). \tag{14.28}$$

Let us again use condition (14.8). We have

$$H\left(y + \frac{b}{\lambda}\right) \geqq H\left(y - \frac{v}{\lambda}\right) e^{-\frac{2b}{\lambda}}; \qquad v \in [-b, b],$$

from which, in view of (14.28), it follows that

$$\varliminf_{y \to +\infty} H\left(y + \frac{b}{\lambda}\right) e^{\frac{2b}{\lambda}} \int_{-b}^{b} \varrho(v)\, dv \geqq 1 - \alpha(b),$$

or

$$\varliminf_{y \to +\infty} H(y) \geqq (1 - \alpha(b))\, e^{-\frac{2b}{\lambda}} \left(\int_{-b}^{b} \varrho(v)\, dv\right)^{-1}. \tag{14.29}$$

Now let $b \to +\infty$ and $\lambda \to +\infty$, so that $b/\lambda \to 0$. Then from (14.29) we obtain the desired inequality (14.25). □

Problem 14.1. Let $N(t)$ be a non-decreasing function, equal to 0 for $t \leqq 1$ and let the integral (14.1) be convergent for $\operatorname{Re} z < -k_0$, some $k_0 > 0$. Assume furthermore, that the function $\zeta(z)$, defined by (14.1), can be meromorphically continued to larger half-space $\operatorname{Re} z < -k_0 + \varepsilon$, where $\varepsilon > 0$ so that on the line $\operatorname{Re} z = -k_0$ there is a single pole at $-k_0$ with principal part $A(z + k_0)^{-l}$ in the Laurent expansion (here l is a positive integer, equal to the order of the pole at $-k_0$). Show that

$$N(t) \sim \frac{(-1)^{l-1} A}{(l-1)!} \cdot t^{k_0} (\ln t)^{l-1}$$

as $t \to +\infty$.

Problem 14.2. Prove the Karamata Tauberian theorem:

Let $N(t)$ be a non-decreasing function of $t \in \mathbb{R}^1$, equal to 0 for $t < 1$ and such that the integral

$$\theta(z) = \int_0^\infty e^{-zt}\, dN(t) \tag{14.30}$$

converges for all $z > 0$ and

$$\theta(z) \sim Az^{-\alpha} \quad \text{as} \quad z \to +0 \tag{14.31}$$

(here $A > 0$ and $\alpha > 0$ are constants). Then

$$N(t) \sim \frac{A}{\Gamma(\alpha+1)}\, t^\alpha \quad \text{as} \quad t \to +\infty\,. \tag{14.32}$$

§15. Asymptotic Behaviour of the Spectral Function and the Eigenvalues (Rough Theorem)

15.1 The spectral function and its asymptotic behaviour on the diagonal. Let M be a closed n-dimensional manifold on which there is given a smooth positive density and let A be a self-adjoint, elliptic operator on M such that

$$a_m(x, \xi) > 0; \qquad \xi \neq 0\,. \tag{15.1}$$

Then A is semibounded. Denote by λ_j its eigenvalues, enumerated in increasing order (counting multiplicities):

$$\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \dots,$$

By $\varphi_j(x)$ we denote the corresponding eigenfunctions, which constitute an orthonormal system.

Let E_t be the spectral projection of A (the orthogonal projection onto the linear hull of all eigenvectors with eigenvalues not exceeding t). It is clear that

$$E_t u = \sum_{\lambda_j \leqq t} (u, \varphi_j)\, \varphi_j\,. \tag{15.2}$$

Definition 15.1. The *spectral function* of A is the kernel (in the sense of L. Schwartz) of the operator E_t.

Taking into account that on M there is a correspondence between functions and densities, we may assume that the spectral function is a function, not a density. From (15.2) it is obvious that this function, $e(x, y, t)$, is given by the formula

$$e(x, y, t) = \sum_{\lambda_j \leqq t} \varphi_j(x)\, \overline{\varphi_j(y)} \tag{15.3}$$