and there is a unique bounded projection $\Pi : \mathcal{H} \to E$ with $\mathrm{Im}\,(\Pi) = E$, $\mathrm{Ker}\,(\Pi) = F$. Then $\Pi$ maps $\mathcal{D}(A)$ into $\mathcal{D}(A)$, and we also see that $\Pi(\mathcal{D}(A))$ is dense in $E$. #

**Lemma 4.21.** *We have*

$$\inf_{u\in\mathcal{D}(A)\cap E,\, \|u\|=1} (Au|u) \le \mu_N(A).$$

**Proof.** We first assume that $\mu_N(A)$ is among the first $N$ eigenvalues (repeated according to their multiplicities) strictly below $\sigma_{\mathrm{ess}}(A)$. Let $e_1, \dots, e_N$ be a corresponding orthonormal family of eigenfunctions. Then $(e_1, \dots, e_N) \cap E \neq \{0\}$ and we let $u = \sum_1^N \lambda_j e_j$ be a normalized vector in the intersection, so that $\sum |\lambda_j|^2 = 1$. Clearly $u \in \mathcal{D}(A) \cap E$, and

$$(Au|u) = \sum_1^N \mu_j |\lambda_j|^2 \le \mu_N \sum_1^N |\lambda_j|^2 = \mu_N.$$

In the case when $\mu_N = \inf \sigma_{\mathrm{ess}}(A)$, the space $1_{]-\infty, \mu_N+\epsilon[}(A)(\mathcal{H}) \subset \mathcal{D}(A)$ is infinite dimensional for every $\epsilon > 0$ and has a non-trivial intersection with $E$. If $u$ is a normalized vector in the intersection, we have $(Au|u) \le \mu_N + \epsilon$, by the spectral theorem, and since $\epsilon > 0$ can be taken arbitrarily small, we obtain the lemma in this case also. #

**Theorem 4.22.** *Under the assumptions above, we have the maxi-min formula:*

$$\mu_N(A) = \sup_{\substack{u_1,\dots,u_{N-1}\in\mathcal{H},\\ \text{linearly independent}}} \inf_{\substack{u\in(u_1,\dots,u_{N-1})^\perp\cap\mathcal{D}(A),\\ \|u\|=1}} (Au|u).$$

**Proof.** We have already seen that the RHS is $\le \mu_N(A)$. When $\mu_N(A)$ is the $N$th eigenvalue $< \inf \sigma_{\mathrm{ess}}(A)$, we can take $u_j = e_j$, $1 \le j \le N-1$ with $e_j$ as in the proof of the last lemma, and see that

$$\inf_{\substack{u\in(e_1,\dots,e_{N-1})^\perp\cap\mathcal{D}(A),\\ \|u\|=1}} (Au|u) = \mu_N(A).$$

The second case $\mu_N(A) = \inf \sigma_{\mathrm{ess}}(A)$ can be treated similarly. #

We end this chapter with a rough determination of the low-lying eigenvalues of semi-classical Schrödinger operators of the type considered in Chapter 3. Further and deeper results in this direction will be given in Chapter 14. Let $M$ be either $\mathbf{R}^n$ or a closed bounded subset of $\mathbf{R}^n$ with smooth boundary and such that 0 belongs to the interior: $0 \in \mathrm{int}\,(M)$. The arguments below will also work with only minor modifications in the case when $M$ is a smooth compact Riemannian manifold, possibly with a boundary. Let $V \in C^\infty(M; [0, \infty[)$ with $V(x) = 0$ precisely for $x = 0$ and assume that $\liminf_{|x|\to\infty} V(x) > 0$, in the case when $M = \mathbf{R}^n$. Let



$$P = -h^2\Delta + V, \tag{4.8}$$

and denote by $P$ also the corresponding Friedrichs extension, when starting from the symmetric operator (4.8) with domain $C_0^\infty(\mathrm{int}\,(M))$. We know that $P \ge 0$ has purely discrete spectrum in $[0, c]$ for some $c > 0$, and we shall now determine the first approximation of the eigenvalues of $P$ in any interval of the form $[0, C_0 h]$ in the limit $h \to 0$.

Let $V_0(x) = \frac{1}{2}\langle V''(0)x, x\rangle$ be the leading term in the Taylor expansion of $V$ at 0, and introduce the harmonic oscillator,

$$P_0 = -h^2\Delta + V_0(x), \ x \in \mathbf{R}^n. \tag{4.9}$$

This operator (realized through the Friedrichs extension) has a purely discrete spectrum, which we essentially computed in Chapter 3, and we saw there that the eigenvalues of $P_0 = P_0(h)$ are of the form $0 < E_0 h < E_1 h \le E_2 h \le \dots$, where $E_j$ are the eigenvalues of $P_0(1)$. A corresponding O.N. basis of eigenfunctions is given by $e_j(x; h) = h^{-n/4} e_j(h^{-1/2}x)$. Here $e_j(x) \in \mathcal{S}(\mathbf{R}^n)$ are Hermite functions up to a linear change of variables.

Let $0 < C_0 \notin \{E_0, E_1, \dots\}$ and let $N_0$ be the number of $E_j$s in $[0, C_0]$, so that $E_{N_0-1} < C_0 < E_{N_0}$.

**Theorem 4.23.** *Under the assumptions above, there exists $h_0 > 0$, such that for $0 < h \le h_0$, $P$ has precisely $N_0$ eigenvalues, $0 \le \lambda_0(h) \le \dots \le \lambda_{N_0-1}(h)$ in $[0, C_0 h]$. Moreover, $\lambda_j(h) = E_j h + \mathcal{O}(h^{3/2})$.*

**Proof.** Let $\chi \in C_0^\infty(\mathrm{int}\,(M))$ be equal to 1 near 0. Since the $L^2$-norm of $(V(x) - V_2(x))\chi(x)e_j(x; h)$ is $\mathcal{O}(h^{3/2})$, we get

$$(P - E_j h)(\chi(x)e_j(x; h)) = r_j(x; h), \ \|r_j\| = \mathcal{O}(h^{3/2}), \tag{4.10}$$

and from this we conclude that for each $j \le N_0 - 1$, there exists an eigenvalue $\mu_j$ of $P$ with $\mu_j = E_j h + \mathcal{O}(h^{3/2})$, but to show that we get $N_0$ eigenvalues of $P$ (counted with multiplicity) in this way, requires a little more work except when the $E_j$ are all distinct.

Let $R \gg 1$ and choose a quadratic partition of unity,

$$\chi_0(x)^2 + \chi_1(x)^2 = 1, \tag{4.11}$$

with $\chi_0, \chi_1 \in C^\infty(M;[0,1])$, $\chi_0 \in C_0^\infty(B(0,2Rh^{1/2}))$, $\chi_0 = 1$ on $B(0,Rh^{1/2})$, $\partial^\alpha \chi_j = \mathcal{O}_\alpha((Rh^{1/2})^{-|\alpha|})$. Here $B(0,r)$ denotes the open ball in $\mathbf{R}^n$ of center 0 and radius $r$. Notice that

$$\chi_0[\Delta,\chi_0] + \chi_1[\Delta,\chi_1] = \chi_0\Delta(\chi_0) + \chi_1\Delta(\chi_1) = \mathcal{O}(\frac{1}{R^2h}), \tag{4.12}$$

which gives the so-called IMS localization formula,

$$\begin{aligned}(-\Delta u|u) &= (-\Delta\chi_0 u|\chi_0 u)\\ &+(-\Delta\chi_1 u|\chi_1 u) + ((\chi_0\Delta(\chi_0) + \chi_1\Delta(\chi_1))u|u),\ u \in \mathcal{D}(P).\end{aligned} \tag{4.13}$$

Combining (4.12), (4.13), we get for $u \in \mathcal{D}(P)$:

$$(Pu|u) = (P\chi_0 u|\chi_0 u) + (P\chi_1 u|\chi_1 u) + \mathcal{O}(\frac{h}{R^2})\|u\|^2. \tag{4.14}$$

Here

$$(P\chi_1 u|\chi_1 u) \geq (V\chi_1 u|\chi_1 u) \geq E_{N_0}h\|\chi_1 u\|^2, \tag{4.15}$$

if we choose first $R$ large enough and then $h$ small enough, so that

$$\inf_{|x|\geq Rh^{1/2}} V \geq E_{N_0}h.$$

On the other hand,

$$(P\chi_0 u|\chi_0 u) = (P_0\chi_0 u|\chi_0 u) + \mathcal{O}(R^3h^{3/2})\|\chi_0 u\|^2, \tag{4.16}$$

when $h$ is small enough, depending on $R$.

Assume that $\chi_0 u \perp e_j(\cdot;h)$, $0 \leq j \leq N_0 - 1$, so that $(P_0\chi_0 u|\chi_0 u) \geq (E_{N_0}h)\|\chi_0 u\|^2$. Then from (4.14)–(4.16), we get

$$(Pu|u) \geq (E_{N_0} - \mathcal{O}(\frac{1}{R^2} + R^3h^{1/2}))h\|u\|^2. \tag{4.17}$$

From the maxi-min principle, it follows that the $(N_0+1)$ st eigenvalue of $P$ is $\geq (E_{N_0} - o(1))h$, when $h \to 0$.

It is easy to find a simple closed loop $\gamma$ in $\{z \in \mathbf{C}; \operatorname{Re} z < C_0 h\}$ such that $\operatorname{dist}(z, \sigma(P) \cup \sigma(P_0(h))) \geq \epsilon_0 h$, $z \in \gamma$ and such that $hE_j$, $j \leq N_0 - 1$ are in the interior of $\gamma$. Here $\epsilon_0 > 0$ is some fixed number independent of $h$. Returning to (4.10), let $E \subset L^2(M)$ be the $N_0$-dimensional space spanned by $\chi e_j(\cdot;h)$, $j = 0,\ldots,N_0-1$, and observe that the functions $\chi e_j$ form an almost O.N. basis in $E$ in the sense that

$$(\chi e_j|\chi e_k) = \delta_{j,k} + \mathcal{O}(h^\infty). \tag{4.18}$$



Rewrite (4.10) as

$$(z-P)(\chi e_j) = (z - E_j h)\chi e_j - r_j,$$

and apply $(z-P)^{-1}(z-E_jh)^{-1}$ for $z \in \gamma$:

$$(z-P)^{-1}\chi e_j = (z-E_jh)^{-1}\chi e_j + (z-P)^{-1}(z-E_jh)^{-1}r_j. \tag{4.19}$$

Let

$$\pi = \frac{1}{2\pi i}\int_\gamma (z-P)^{-1}dz$$

be the spectral projection associated with $P$, and the intersection of $\mathbf{R}$ and the interior of $\gamma$ and let $F = \pi(L^2(M))$. From the conclusion after (4.17), we know that $\dim F \leq N_0$. From (4.19) we get

$$\pi(\chi e_j) = \chi e_j + k_j,\ \|k_j\| = \mathcal{O}(h^{1/2}), \tag{4.20}$$

and it follows (when $h > 0$ is small enough) that the dimension of $F$ is at least equal to $N_0$, so we have finally $\dim F = N_0$. Moreover, $f_j = \pi(\chi e_j)$, $0 \leq j \leq N_0 - 1$ form a basis in $F$, and if we use (4.10) again, we get $Pf_j = hE_j f_j + \mathcal{O}(h^{3/2})$. Let $f = (f_0,\ldots,f_{N_0-1})$ be the corresponding row vector and introduce the orthonormalized basis $g = f((f_j|f_k))^{-1/2}$. The matrix of $P_{|F}$ is then $\operatorname{diag}(hE_j) + \mathcal{O}(h^{3/2})$. If $\lambda_0 \leq \ldots \leq \lambda_{N_0-1}$ denote the eigenvalues of $P_{|F}$, it follows that $\lambda_j = hE_j + \mathcal{O}(h^{3/2})$. This completes the proof of the Theorem. #

Let $E_j$ be one of the eigenvalues of the harmonic oscillator $P_0(1)$ with $j \leq N_0 - 1$, and assume that $E_j$ is a simple eigenvalue. It follows from Theorem 3.6 that the corresponding eigenvalue $\lambda_j(h)$ has an asymptotic expansion $\sim h(a_0 + a_1h + a_2h^2 + \ldots)$, where $a_0 = E_j$. If we drop the assumption that $E_j$ is simple then we still have an asymptotic expansion for $\lambda_j(h)/h$ with leading term $E_j$, provided that we allow for half powers of $h$. See Simon [Si1], Helffer–Sjöstrand [HeSj1].

## Notes

Most of this chapter is a compilation of general and well-known facts for spectral theory. We have used [NaRi], [ReSi] and [CFKS]. The asymptotic behavior of the lowest eigenvalues of the Schrödinger operator in the semi-classical regime has been studied by many authors. See [Si1], [HeSj1]. In the one-dimensional case precise asymptotic expansions were computed by Combes, Duclos and Seiler [CDS]. The case of the asymptotic behavior when the minimum is degenerate has been studied by Martinez and Rouleux [MR].