Ikehara's Tauberian theorem and the prime number theorem (Flemming's talk)

based on Shubin §14 and Zagier's article on the prime number theorem,

Thm (Ikehara): $N(t)$ increasing function on $\mathbb{R}$, $N(t)=0$ for $t\le 1$.

If $\zeta(z) := \int_1^\infty t^{-z}\, dN(t)$, $dN(t) = d\mu(t)$ where $N(x) = \mu((-\infty, x])$

$= \langle t^{-z}, \partial_t N\rangle$

converges for $\operatorname{Re} z > k_0 > 0$ and $\zeta(z) - \frac{A}{z-k_0}$ is continuous for $\operatorname{Re} z \ge k_0$, with $A \neq 0$, then $N(t) \sim \frac{A}{k_0} t^{k_0}$.

The latter means that $\lim_{t\to\infty} \frac{N(t)}{\frac{A}{k_0} t^{k_0}} = 1$.

Simplifications: $\zeta_1(z) = \zeta(k_0 z) \Rightarrow \zeta(k_0 z) - \frac{A}{k_0 z - k_0} = \zeta_1(z) - \frac{A/k_0}{z-1}$, $A/k_0 =: A_1$

$\Rightarrow$ May assume $k_0 = 1$.

As $\zeta_1(z) \ge 0$, $z - 1 \ge 1$ for $z \ge 1$ $\Rightarrow$ $A_1 > 0$.

Replacing N by $N_1(t) = A^{-1} N(t)$, we may assume $A_1 = 1$.

So let now $k_0 = A = 1$.

Consider $\varphi(x) = N(e^x) \Rightarrow \zeta(z) = \int_0^\infty e^{-tz}\, d\varphi(t)$

$N(t) \sim t \iff \varphi(x) \sim e^x \iff \varphi(x) e^{-x} \xrightarrow{x\to\infty} 1$.

$H(x) := \varphi(x) e^{-x}$, $\varphi(x)$ increasing $\iff H(y) \ge H(x) e^{x-y}$ $\forall y \ge x$.

$\zeta(z) = \int_0^\infty e^{-tz}\, d\varphi(t) = \lim_{x\to\infty} e^{-xz}\varphi(x) - e^{-0z}\varphi(0) + z\int_0^\infty e^{-tz}\varphi(t)\, dt$

As $\operatorname{Re} z > 1$, the boundary terms disappear.

$\overset{\varphi}{=} z \int_0^\infty e^{-tz} \varphi(t)\, dt$

$= z \int_0^\infty e^{-t(z-1)} H(t)\, dt = z \int_0^\infty e^{-t(z-1)} (H(t)-1)\, dt + \frac{z}{z-1}$

Note that $\frac{\zeta(z)}{z} - \frac{1}{z-1} = \frac{1}{z}\left(\zeta(z) - \frac{1}{z-1} - 1\right)$.

$z = 1 + \varepsilon + it$, $\int_0^\infty e^{-x\varepsilon - xit}(H(x) - 1)\, dx =: h_\varepsilon(t)$.

Thm: Let $H(x)$ be 0 for $x \leq 0$, $H(y) \geq H(x)\, e^{x-y}$ for $y \geq x$.

If $h_\varepsilon(t)$ converges absolutely for $\varepsilon > 0$ and $h(t) = \lim_{\varepsilon \to 0+} h_\varepsilon(t)$ uniformly for $|t| \leq \lambda$, $\lambda > 0$ arbitrary, then $\lim_{x \to \infty} H(x) = 1$.

By the above, this theorem implies Ikehara's thm.

Lemma: Let $\rho \in L^1(\mathbb{R})$ s.t. $\rho(x) \geq 0$ a.e., $\hat\rho(0) = 1$, $\hat\rho \in C_c(\mathbb{R})$. Then for any fixed $\lambda > 0$ we have $\lim_{y \to \infty} \int_{-\infty}^{y} H(y - \frac{x}{\lambda})\, \rho(x)\, dx = 1$

Example: $\int_{-2}^{2} e^{ity} \left(1 - \frac{|t|}{2}\right) dt = \frac{2 \sin^2(y)}{y^2}$ will give such a ρ.

$(FG)^\wedge = \hat F * \hat G$

Proof of Lemma: $\frac{1}{2\pi} \int e^{ity}\, h_\varepsilon(t)\, \hat\rho(t)\, dt = \int e^{-x\varepsilon} (H(x) - 1)\, \rho(y-x)\, dx$

or $\frac{1}{2\pi} \int e^{ity}\, h_\varepsilon(t)\, \hat\rho(t)\, dt + \int e^{-x\varepsilon} \rho(y-x)\, dx = \int e^{-x\varepsilon} H(x)\, \rho(y-x)\, dx$

$\varepsilon \downarrow 0$: $\frac{1}{2\pi} \int e^{ity}\, h(t)\, \hat\rho(t)\, dt + \underbrace{\int \rho(y-x)\, dx}_{=1} = \int H(x)\, \rho(y-x)\, dx$

$y \to \infty$: Riemann–Lebesgue: 1^{st} summand on left $\to 0$

$\Longrightarrow \lim_{y \to \infty} \int H(x)\, \rho(y-x)\, dx = 1.$

This proves the Lemma for $\lambda = 1$.

General case: $\int H(y - \frac{x}{\lambda})\, \rho(x)\, dx = \int H(y-x)\, \lambda \rho(\lambda x)\, dx$

$\rho_\lambda(x) = \lambda \rho(\lambda x)$ satisfies all the assumptions on ρ

$\Longrightarrow$ assertion for all λ $\square$

Proof of theorem: $\limsup_{x \to \infty} H(x) \leq 1$: Note that $H(y - \frac{x}{\lambda}) \geq H(y - \frac{a}{\lambda})\, e^{-\frac{2a}{\lambda}}$ whenever $x \in [-a, a]$.

So $\limsup_{y \to \infty} H(y - \frac{a}{\lambda})\, e^{-\frac{2a}{\lambda}} \int_{-a}^{a} \rho(x)\, dx \leq \limsup_{y \to \infty} \int_{-a}^{a} H(y - \frac{v}{\lambda})\, \rho(v)\, dv$

(Lemma) ≤ 1

$\lambda = a^2$: $\limsup_{y \to \infty} H(y) \leq e^{2/a} \left(\int_{-a}^{a} \rho(v)\, dv \right)^{-1} \longrightarrow 1.$

$\liminf_{y\to\infty} H(y) \geq 1$: $\quad \int_{|v|\geq b} H(y-\frac{v}{\lambda})\, p(v)\, dv \leq a(b) \to 0$ uniformly

So $\int_{-b}^{b} H(y-\frac{v}{\lambda})\, p(v)\, dv \geq 1 - a(b)$.

Since $H(y+\frac{b}{\lambda}) \geq H(y+\frac{v}{\lambda})\, e^{-2b/\lambda}$ for $v\in[-b,b]$, we get that

$$\Rightarrow H(y+\tfrac{b}{\lambda})\, e^{+2b/\lambda} \int_{-b}^{b} p(v)\, dv \geq 1 - a(b)$$

$$\Rightarrow \liminf_{y\to\infty} H(y+\tfrac{b}{\lambda}) \geq (1-a(b))\, e^{-2b/\lambda} \left(\int_{-b}^{b} p(v)\, dv\right)^{-1} \to 1$$

$\lambda = b^2,\ b\to\infty$ ▫

Prime number theorem:

$$\{\text{Number of primes} \leq t\} \sim \frac{t}{\log(t)}$$

Riemann's ζ-fct. : $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum 2^{-n_1 s}\, 3^{-n_2 s} \ldots$

$$= \prod_{p \text{ prime}} \sum_{n=0}^{\infty} p^{-ns} = \prod_{p \text{ prime}} (1-p^{-s})^{-1} \quad \text{(geometric series)}$$

Note that $\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} x^{-s}\, dx = \sum_{n=1}^{\infty} \int_n^{n+1} \left[\frac{1}{n^s} - \frac{1}{x^s}\right] dx \quad (*)$

Furthermore $\left|\int_n^{n+1} [n^{-s} - x^{-s}]\, dx\right| = \left|\int_n^{n+1} s \int_n^{x} u^{-s-1}\, du\, dx\right|$

$$\leq \max_{n\leq u\leq n+1} \frac{|s|}{|u^{s+1}|} = \frac{|s|}{n^{\operatorname{Re} s+1}}$$

So $(*)$ converges for $\operatorname{Re} s > 0$ and extends $\zeta(s)$ to this region.

Look at $-\frac{\zeta'(s)}{\zeta(s)} = \prod_p (1-p^{-s}) \sum_q \frac{1}{(1-q^{-s})^2} \log q\; q^{-s} \prod_{p\neq q} (1-p^{-s})^{-1}$

$$= \sum_q \frac{\log q\; q^{-s}}{1-q^{-s}} = \sum_q \frac{\log q}{q^s - 1} = \sum_q \frac{\log q}{q^s} - \sum_q \frac{\log q}{q^s(q^s-1)}$$

Note that $\sum_q \frac{\log(q)}{q^s(q^s-1)}$ continuous for $\operatorname{Re} s > \frac{1}{2}$.

Show that $\zeta(1+i\alpha) \neq 0 \quad \forall \alpha \in \mathbb{R}$. Assume not. Then, using $\overline{\zeta(\bar{s})} = \zeta(s)$, both $1+i\alpha_0$ and $1-i\alpha_0$ are zeros of ζ.

Assume they are of order μ. Let ν be the order of the zero in $1+2i\alpha_0$ (if there is one), and $\nu=0$ if $\zeta(1+2i\alpha_0) \neq 0$. Then $-\lim_{s\to 1} (s-1)\frac{\zeta'(s)}{\zeta(s)} = 1$

and $\lim_{s\to 1+i\alpha} -(s-1-i\alpha_0)\frac{\zeta'(s)}{\zeta(s)} = -\mu$

$\lim_{s\to 1+2i\alpha} -(s-1-2i\alpha_0)\frac{\zeta'(s)}{\zeta(s)} = -\nu$

Define $\phi(s) = \sum_q \frac{\log(q)}{q^s}$. $\sum_{r=-2}^{2} \binom{4}{2+r} \phi(1+\varepsilon + ir\alpha_0) =$

$= \sum_q \frac{\log(q)}{q^{1+\varepsilon}} \sum_{r=-2}^{2} \binom{4}{2+r} q^{\mp ir\alpha_0} \overset{\text{binomial theorem}}{=} \sum_q \frac{\log(q)}{q^{1+\varepsilon}} \left(q^{-\frac{i\alpha_0}{2}} + q^{+\frac{i\alpha_0}{2}}\right)^4 \geq 0$

Then $0 \leq \lim_{\varepsilon\to 0^+} \varepsilon \sum_{r=-2}^{2} \binom{4}{2+r} \phi(1+\varepsilon \pm ir\alpha_0) = \dots = -\nu - 4\mu + 6 - 4\mu - \nu$

$= 6 - 8\mu - 2\nu \leq -2$ (μ, ν integers) ↯

<u>Conclusion</u>: $\zeta(1+i\alpha) \neq 0 \quad \forall \alpha \in \mathbb{R}$.
$\phi(s) - \frac{1}{s-1}$ continuous for $\operatorname{Re} s \geq 1$.

<u>Final step</u>: Apply Ikehara to $N(t) = \sum_{\substack{p\leq t\\ \text{prime}}} \log(p)$ $(\Rightarrow \zeta(z) = \phi(z))$

Ikehara implies that $N(t) \sim t$ for $t \to \infty$.

On the other hand $N(t) = \sum_{\substack{p\leq t\\ \text{prime}}} \log(p) \leq \sum_{p\leq t} \log(t) \leq \{\text{number of primes} \leq t\} \log(t)$

and $N(t) \geq \sum_{\substack{t^{1-\varepsilon}\leq p\leq t\\ \text{prime}}} \log(p) \geq \sum_{\substack{t^{1-\varepsilon}\leq p\leq t\\ \text{prime}}} (1-\varepsilon)\log(t) = (1-\varepsilon)\log(t)\left(\sum_{\substack{p\leq t\\ \text{prime}}} 1 - \sum_{\substack{p\leq t^{1-\varepsilon}\\ \text{prime}}} 1\right)$

$\geq (1-\varepsilon)\log(t)\left(\{\text{number of primes} \leq t\} - t^{1-\varepsilon}\right) \quad \forall \varepsilon > 0$. ∎

$\Rightarrow \forall \varepsilon > 0: \frac{t}{\log(t)} \sim \frac{N(t)}{\log(t)} \leq \{\text{number of primes} \leq t\} \leq \frac{N(t)}{(1-\varepsilon)\log(t)} + (1-\varepsilon)\log(t)\, t^{1-\varepsilon} \sim \frac{t}{(1-\varepsilon)\log(t)}$