Propagation of singularities

$A = op(a) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ linear DO, $a_\alpha \in C^\infty$, a_α realvalued for $|\alpha| = m$.

$u \in S'$ solution to $Au = f \in C^\infty$.

Recall $WF(Au) \subseteq WF(u) \subseteq WF(Au) \cup \text{char}\, A$, $\text{char}\, A = \{(x,\xi) : \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha = 0\}$

We are now going to study $WF u$ more precisely, showing that it is a union of "bicharacteristic curves".

Recall that one may identify a vectorfield with a first-order DO, the directional derivative in the direction of the vector field: $\begin{pmatrix} V_1(x) \\ \vdots \\ V_n(x) \end{pmatrix} \longleftrightarrow \sum V_i(x)\partial_{x_i}$

Def: Let $p(x,\xi) \in C^\infty$ real valued.

$$H_p := \sum_{i=1}^n \left(\frac{\partial p}{\partial \xi_i} \partial_{x_i} - \frac{\partial p}{\partial x_i} \partial_{\xi_i} \right) \quad \text{Hamiltonian vector field}$$

(From physics classes on Classical Mechanics, you might know the Poisson bracket $\{q, p\} := H_p q$)

Remark: H_p is tangent to $P_c := \{(x,\xi) : p(x,\xi) = c\}$, $c \in \mathbb{R}$.

Proof: The normal to P_c is $\nabla p = (\partial_x p, \partial_\xi p)$.

$H_p \longleftrightarrow (\partial_\xi p, -\partial_x p) \perp \nabla p$, so H_p is orthogonal to the normal to P_c, i.e. tangent to P_c $\square$

Def: a) $(x(t), \xi(t))_{t \in (-\varepsilon, \varepsilon)}$ integral curve of H_p if

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} \longleftrightarrow H_p \quad \text{i.e.} \quad \begin{aligned} \frac{d}{dt} x(t) &= \partial_\xi p(x(t), \xi(t)) \\ \frac{d}{dt} \xi(t) &= -\partial_x p(x(t), \xi(t)) \end{aligned} \quad (*)$$

b) $(x(t), \xi(t))_{t \in (-\varepsilon, \varepsilon)}$ bicharacteristic curve of p if $(x(t), \xi(t))$ is an integral curve of H_p and $p(x(t), \xi(t)) = 0 \ \forall t$.

Remark: a) If $(x(t),\xi(t))$ is an integral curve, $p(x(t),\xi(t))$ is constant:

$$\frac{d}{dt}\, p(x(t),\xi(t)) = \partial_x p(x(t),\xi(t))\, \frac{d}{dt} x(t) + \partial_\xi p(x(t),\xi(t))\, \frac{d}{dt}\xi(t)$$

$$= \partial_x p \;\; \partial_\xi p - \partial_\xi p\; \partial_x p = 0 .$$

In physics, the trajectory $(x(t),\xi(t))$ of the coordinate x and the momentum ξ in phase space is the bicharacteristic of the Hamiltonian ($\approx$ energy).

b) $(*)$ always has a unique solution by general existence theorems for ordinary differential equations (Picard-Lindelöf)

c) On $\{(x,\xi) : p(x,\xi)=0\}$ we have for any q:

$$H_{pq} = \sum \underbrace{\frac{\partial(pq)}{\partial \xi_i}}_{\overset{!}{=}(\partial_{\xi_i} p)q + \underset{\overset{\|}{0}}{p}(\partial_{\xi_i} q)} \partial_{x_i} - \underbrace{\frac{\partial(pq)}{\partial x_i}}_{=(\partial_{x_i} p)q + \underset{\overset{\|}{0}}{p}(\partial_{x_i} q)} \partial_{\xi_i} = q\, H_p \quad \text{parallel to } H_p$$

$\Rightarrow$ up to reparametrization, the bichar. curves of p and pq are locally the same.

Thm: (Propagation of Singularities)

Let $A = a(x,D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$, a_α real valued for $|\alpha|=m$, and $u \in S'$ s.t. $Au = f \in C^\infty$ $\Longrightarrow$ WFu is a union of bicharacteristic curves of $\sum_{|\alpha|=m} a_\alpha \xi^\alpha$.

Example: For the wave equation $\partial_{x_1}^2 u - \sum_{j=2}^{n} \partial_{x_j}^2 u = f$,

the bicharacteristic curves are

$$\begin{pmatrix} x_1(t) \\ x'(t) \\ \xi_1(t) \\ \xi'(t) \end{pmatrix} = \begin{pmatrix} \tau t + x_1(0) \\ -\eta t + x'(0) \\ \tau \\ \eta \end{pmatrix}, \quad x_1(0), x'(0), \tau, \eta \text{ constants}, \quad |\tau| = |\eta| .$$

So the wave front set propagates along straight lines in the light cone $\left| x'(t) + \frac{\eta}{\tau} x_1(t) \right| = \text{const}$.

Notation: $C^0(I, H^k) = \{ w : I \to H^k \text{ continuous} \}$, $I \subseteq \mathbb{R}$ compact interval,

Banach space with norm $\|w\|_{I,k} := \sup_{t \in I} \|w(t)\|_{H^k}$

Similarly: $C^m(I, H^k)$, $C^m(I, S)$.

Lemma 4.3: $b \in S^1$ (symbol of order 1), $b - \bar{b} \in S^0$, $I = [-s, s]$ for some $s > 0$
(op(b) properly supported)

$$\Longrightarrow \forall k \in \mathbb{Z} : \exists \mu = \mu(k) \ \forall w \in C^0(I; H^{k+1}) \cap C^1(I, H^k)$$

(Energy estimates)
$$\begin{cases} \sup_{t \in I} \|e^{\mu t} w(t)\|_{H^k} \leq \|e^{\mu s} w(s)\|_{H^k} + 2 \int_I \|e^{\mu t} (\partial_t - i\, op(b)) w(t)\|_{H^k} dt \\ \sup_{t \in I} \|e^{-\mu t} w(t)\|_{H^k} \leq \|e^{\mu s} w(-s)\|_{H^k} + 2 \int_I \|e^{-\mu t} (\partial_t - i\, op(b)^*) w(t)\|_{H^k} dt \end{cases}$$

Moreover: $\forall g \in C^0(I, S) \ \forall \chi \in S \quad \exists!\ w \in \bigcup_k C^0(I, H^k)$ s.t.

$$\begin{cases} (\partial_t - i\, op(b)) w(t) = g(t) \\ w(s) = \chi \end{cases}$$

w even belongs to $\bigcap_k C^0(I, H^k)$.

Proof: 1.) Energy estimates

$$|\mathrm{Re}\,(e^{\mu t} i\, op(b) w(t), e^{\mu t} w(t))| = \tfrac{1}{2} |(i \underbrace{(op(b) - op(b)^*)}_{\text{order } 0} e^{\mu t} w(t), e^{\mu t} w(t))|$$

$$\leq C \|e^{\mu t} w(t)\|^2_{L^2}$$

Let $\mu \geq C \to \frac{d}{dt} \|e^{\mu t} w(t)\|^2_{L^2} = 2\mu \|e^{\mu t} w(t)\|^2_{L^2} + 2\,\mathrm{Re}(e^{\mu t} \partial_t w, e^{\mu t} w)$

$$\geq \underbrace{2(\mu - C) \|e^{\mu t} w(t)\|^2_{L^2}}_{\geq 0} + 2\,\mathrm{Re}(e^{\mu t} (\partial_t - i\, op(b)) w, e^{\mu t} w)$$

$$\geq -2 \underbrace{\|e^{\mu t} (\partial_t - i\, op(b)) w(t)\|_{L^2}}_{=: Z(t)} \underbrace{\|e^{\mu t} w(t)\|_{L^2}}_{=: W(t)}$$

$\sup_{t \in I} \int_t^s \dots \Rightarrow \sup_{t \in I} (W(s)^2 - W(t)^2) \geq -2 \int_I Z(t)\, dt \ \sup_{t \in I} W(t)$

$$\Longrightarrow W(s)^2 + 2 (\sup W) \int_I Z \geq (\sup W)^2$$

or $(\sup W - \int Z)^2 \leq W(s)^2 + (\int Z)^2 \leq (W(s) + \int Z)^2$

$\Rightarrow$ 1st Energy estimate with $k = 0$, 2nd estimate: $t \leftrightarrow -t$, $op(b) \leftrightarrow op(b)^*$

$\underline{k \neq 0:}$ $\Lambda_k \approx op\langle\xi\rangle^k$ properly supported, $\therefore \Lambda_{-k}\Lambda_k = 1 + r$, $r \in S^{-\infty}$

$$\Lambda_k\,(\partial_t - i\,op(b))w = \left(\partial_t\Lambda_k - i\,\Lambda_k\,op(b)(\Lambda_{-k}\,\Lambda_k - r)\right)w$$

$$\overset{!}{=} \Big(\partial_t - i\underbrace{\big(\underbrace{\Lambda_k\,op(b)\,\Lambda_{-k}}_{\in S^1} + \underset{S^{-\infty}}{\underset{\wedge}{\tilde r}}\big)}_{b_k}\Big)\,\Lambda_k w$$

b_k still properly supported, $b_k - \overline{b_k} \in S^0$. Apply previous result with b_k instead of b, $\Lambda_k w$ instead of w.

2.) Uniqueness

Given solutions w_1, w_2 with same data g, χ, let $w := w_1 - w_2 \in C^0(I, H^{k+1})$. w satisfies the homogeneous equation

$$\partial_t w - i\,op(b)w = 0, \quad w(s) = 0,$$

$w \in C^0(I, H^{k+1}) \Rightarrow \partial_t w = i\,op(b)w \in C^0(I, H^k) \Rightarrow w \in C^1(I, H^k)$

Part 1.) $\Rightarrow \sup_{t\in I} \|e^{rt}w(t)\|_{H^k} \leq 0 \Rightarrow w(t) = 0 \Rightarrow w_1 = w_2$

3.) Existence and Smoothness:

Show: $\forall g \in C^0(I, S), \chi \in S, k \in \mathbb{Z}\ \exists$ solution $w_k \in C^0(I, H^{k-1})$

(Then 2.) $\Rightarrow$ all the w_k's are equal $\rightarrow w = w_{k+1} \in \bigcap_k C^0(I, H^k)$)

Let $\mathbb{E} := \{(\partial_t - i\,op(b)^*)\psi \ :\ \psi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^n) : \operatorname{supp}\psi \subseteq \{(t,x) : t > -s\}\}$.

$$\begin{cases} (\partial_t - i\,op(b)^*)\Psi = \varphi \in \mathbb{E} \\ \Psi(-s) = 0 \end{cases}$$ has solution $\Psi \in C^0(I, S)$, namely the restriction of ψ from the def. of $\mathbb{E}$ to I. 2.) $\Rightarrow \Psi$ unique solution

Energy estimate $\sup_{t\in I} \|\psi(t)\|_{H^{-k}} \leq C_k \int_I \|\varphi(t)\|_{H^{-k}}\,dt \quad \forall k \in \mathbb{Z}$ $(\square)$

For $g \in C^0(I, S), \chi \in S$ define
$$W : \mathbb{E} \longrightarrow \mathbb{C}$$
$$\varphi \longmapsto (\chi, \Psi(s)) - \int_I (g(t), \Psi(t))\,dt$$

W continuous in $L^1(I, H^{-k})$-norm: $|W(\varphi)| \leq \left(\|\chi\|_{H^k} + \int_I \|g\|_k\right) \sup_{t\in I} \|\psi\|_{H^{-k}} \overset{(\square)}{\leq} C_k' \int_I \|\varphi\|_{H^{-k}}$

$\Longrightarrow \exists\, w \in L^\infty(I, H^k) \cong L^1(I, H^{-k})^* : W(\varphi) = (w, \varphi) \overset{\text{def of }\varphi}{=} \int_I (w(t), (\partial_t - op(b)^*)\Psi)\,dt$

If $\operatorname{supp} \tilde{\psi} \subseteq \{(t,x) : |t|<s\} \Rightarrow (\chi, \tilde{\psi}(s)) = 0$

$$\Rightarrow \int_I (g, \tilde{\psi}) = \int_I (w, (\partial_t - i\operatorname{op}(b))^* \tilde{\psi})$$

$\xRightarrow{\forall \tilde{\psi}}$ $(\partial_t - i\operatorname{op}(b))\, w = g$ in $\{(t,x) : |t|<s\}$

$\Rightarrow \partial_t w \in L^\infty(I, H^{k-1}) \rightsquigarrow w \in C^0(I, H^{k-1})$

Analogous: $w \in C^1(I, H^{k-2})$.

Given $\varphi \in C_c^\infty(\mathbb{R}^n) \Rightarrow \exists \psi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$, $\operatorname{supp}\psi \subseteq \{(t,x) : t > -s\}$, $\psi(s) = \varphi$

s.t. $\int_I (\underset{\downarrow}{w(t)}, \underset{\uparrow}{\partial_t \psi})\, dt = (w(s), \varphi) = (\chi, \varphi)$ $\rightsquigarrow$ w is the desired solution. $\square$

Remark:
- $t \leftrightarrow -t$: Can also solve the initial value problem

$$(\partial_t - i\operatorname{op}(b))\, w = g, \quad w(-s) = \chi.$$

- Proof is easily modified to include t-dependent symbols b.

Proof of Thm:

Note: It is enough to show $WF(u) \cap \gamma \subseteq \gamma$ open $\forall$ bicharacteristic γ.

⌈Proof: $WF(u)$ closed, γ connected $\Rightarrow WF(u) \cap \gamma \subseteq \gamma$ closed ⌋

Show: $(x_0, \xi_0) \in WF(u) \subseteq \operatorname{char} A \Rightarrow \exists$ open neighborhood U s.t. the bichar. $\gamma = \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}$ starting at $\begin{pmatrix} x_0 \\ \xi_0 \end{pmatrix}$ satisfies $\gamma \cap U \subseteq WF(u) \cap U$.

Assume that this was false. If $\psi \in C_c^\infty$, $\psi \equiv 1$ near x_0, the operator $Bu := \Lambda_{m-1}(\psi u)$ has principal symbol $|\xi|^{m-1}\psi(x) \neq 0$ in an open neighborhood U of (x_0, ξ_0), so that

$$WF(Bu) \cap U \subseteq WF(u) \cap U \subseteq (WF(Bu) \cup \operatorname{char} B) \cap U$$
$$= WF(Bu) \cap U \quad (\operatorname{char} B \cap U = \emptyset)$$

So we may assume that $\gamma \cap U \not\subseteq WF(Bu) \cap U \;\forall u$.

Advantages:
- $\psi u \in \mathcal{E}'$, so $\operatorname{op}(r)\, Bu \in C^\infty_{comp}$ $\forall r \in S^{-\infty}_{\text{prop supported}}$
- $\varphi \equiv 1$ near x_0, $\psi \equiv 1$ on $\operatorname{supp}\varphi \Rightarrow \varphi A \psi u = \varphi A u$, $= \varphi f$

  so $\underbrace{(\varphi A \Lambda_{1-m})}_{Q \in \operatorname{Op}(S^1)} Bu = \varphi f$, and $Q = \operatorname{op}(q)$ has the same bicharacteristic curves near (x_0, ξ_0) as A.

Goal: $(x_0,\xi_0) \notin WF(\underbrace{QBu}_{=:v})$ $(\Leftrightarrow (x_0,\xi_0) \notin WF(u))$.

Idea: Construct $c_0 \in S^0$ elliptic near (x_0,ξ_0) s.t. $op(c_0)v \in C^\infty$.
($\Rightarrow$ assertion because then $(x_0,\xi_0) \notin WF(op(c_0)v) \cup \text{char}\, op(c_0) \supseteq WF(v)$)

To do so, construct $c(t,x,\xi)$ s.t. $c(0,t,x) = c_0(x,\xi)$
and $w(t) := op(c(t,x,\xi))v$ satisfies Cauchy problem from Lemma 4.9.
Since $w \in C^0(I,H^N)$ for some $N \overset{4.9}{\Longrightarrow} w \in \bigcap_k C^0(I,H^k)$
and hence $op(c_0)v = w(0) \in C^\infty$.

Construct: $c(t,x,\xi) \in C^1(I,S^0)$ s.t. $r(t) := \partial_t c - i(q_1\#c - c\#q_1)$
$\in C^0(I,S^{-\infty})$.

Let $c_0 \in S^0$ homogeneous for $|\xi| \geq 1$ of degree 0. Ansatz: $c \sim \sum c_{-j}$,
c_{-j} hom of degree $-j$:

$$0 \sim \sum_j \partial_t c_{-j} - i \sum_j (\underbrace{q\#c_{-j} - c_{-j}\#q}_{\text{principal symbol: } \frac{1}{i}\left(\frac{\partial q_1}{\partial\xi}\frac{\partial c_{-j}}{\partial x} - \frac{\partial q_1}{\partial x}\frac{\partial c_{-j}}{\partial \xi}\right) = \frac{1}{i}H_{q_1}c_{-j}}))$$

order 0: $0 = \partial_t c_0 - H_{q_1}c_0$.

order -1: $0 = \partial_t c_{-1} - H_{q_1}c_{-1} - \underbrace{\text{Junk}(q_1, q_0, c_0)}_{\text{known}}$

i.e. $\partial_t c_{-1} - H_{q_1}c_{-1} = \text{Junk}(q_1, q_0, c_0)$.

$\vdots$

order $-j$: $\partial_t c_{-j} - H_{q_1}c_{-j} = \text{Junk}(q_1, \ldots, q_{-j+1}, c_0, \ldots, c_{-j+1})$

ODE-existence theorems (Picard-Lindelöf): For each of the above equations $\exists$ solution c_{-j} in a neighborhood of (x_0,ξ_0).

As all the equations are homogeneous in $|\xi| \geq 1$, c_{-j} homog. of degree $-j$. $\text{Supp}\, c_{-j}(t) \subseteq \text{Supp}\, c_0(t)$.

$\text{supp}\, c_0(t) \subseteq$ Time-t-flow along integral curves of H_q of points in supp
in supp $c_0(0)$.

Choosing supp c_0 in a small conic neighborhood Γ of (x_0, ξ_0).

Homogeneity easily implies $c_0(t) \in S^0$, if $c_0(t)$ is just smooth for $|\xi| = 1$ (which it is). Similarly for $c_j(t)$. Let $c \sim \sum c_j$ prop. supp.

By assumption $\gamma \cap U \notin WF(v)$, so $\exists\, s$: $\begin{pmatrix} x(s) \\ \xi(s) \end{pmatrix} \notin WF(v)$ for some s s.t. c is defined.

If supp c_0 was a small neighborhood of (x_0, ξ_0) $\Rightarrow$ supp $c_0(s)$ small neighborhood of $(x(s), \xi(s))$ $\Rightarrow$ supp $c(s)$ small neighborhood of $(x(s), \xi(s))$

$\Longrightarrow$ $w(s) := op(c(s))\, v = \chi \in C^\infty$.

But $\varphi u \in \mathcal{E}' \Rightarrow \varphi u \in H^{-N}$ for some N $\Rightarrow v \in H^{-N}$ for some N

$$\Longrightarrow \quad w(t) = op(\underbrace{c(s)}_{\in C^0(I,\, S^0)})\, v \in C^0(I, H^{-N})$$

$$\partial_t w - i\, op(b) w = op(r(t, x, \xi))\, v - i\, op(c(t, x, \xi))\, op(b(x, \xi))\, v$$

by construction of c. As all operators are properly supported, the right hand side $\in C^0(I, S)$, even $C^0(I, C_c^\infty)$.

$\Longrightarrow$ $w \in C^0(I, H^{-N})$ solution of PDE in Lemma 4.3

$\Longrightarrow$ $w \in \bigcap_k C^0(I, H^k)$ $\Rightarrow$ $w(0) = op(c_0(x, \xi))\, v \in H^\infty \subset C^\infty$.

$\Longrightarrow$ $(x_0, \xi_0) \notin WF(v)$ $\rightarrow$ $(x_0, \xi_0) \notin WF(u)$, which is what we wanted to show. $\square$