Elements of spectral theory for differential operators

Recall some basic facts about multiplication operators and the spectral theorem:

Let (X,μ) be a measure space, $f: X\to\mathbb{R}$ measurable,
$M_f: D(M_f)\subset L^2(\mu)\longrightarrow L^2(\mu)$, defined by $M_f\psi(x)=f(x)\psi(x)$
for $\psi\in D(M_f):=\{\varphi\in L^2(\mu): f\varphi\in L^2(\mu)\}$.

<u>Thm:</u> a) M_f densely defined, self-adjoint.

b) $\operatorname{spec}(M_f)=\operatorname{ess\,ran} f:=\{\lambda\in\mathbb{R}: \forall\varepsilon>0: \mu(\{x: |f(x)-\lambda|<\varepsilon\})>0\}$

<u>Proof:</u> a) Let $X_n:=\{x\in X: |f(x)|\le n\}\Rightarrow \bigcup_n X_n=X$,

so $\bigcup_n\{\psi\in L^2(\mu): \operatorname{supp}\psi\subseteq X_n\}$ dense in $L^2(\mu)$

and $\bigcup_n\{\dots\}\subset D(M_f)$.

M_f symmetric: $\langle M_f\varphi,\psi\rangle_{L^2}=\langle\varphi,M_f\psi\rangle_{L^2}\quad\forall\varphi,\psi$

$\Rightarrow M_f\subset M_f^*$

<u>$M_f^*\subset M_f$</u>: Let $\psi\in D(M_f^*)\Rightarrow \overbrace{\chi_{\Omega_n}}^{\in L^\infty}M_f^*\psi\in L^2(\mu)$

$\underbrace{\chi_{\Omega_n}f}_{\in L^\infty}\cdot\psi\in L^2(\mu)$

$\Rightarrow\forall\varphi\in D(M_f): \langle\varphi,\chi_{\Omega_n}M_f^*\psi\rangle=\langle\chi_{\Omega_n}\varphi,M_f^*\psi\rangle$

$=\langle M_f(\chi_{\Omega_n}\varphi),\psi\rangle$

$=\langle f\chi_{\Omega_n}\varphi,\psi\rangle=\langle\varphi,f\chi_{\Omega_n}\psi\rangle$

$\Rightarrow \chi_{\Omega_n}M_f^*\psi=f\chi_{\Omega_n}\psi$ a.e. $\forall n\Rightarrow M_f^*\psi=f\psi$ a.e.

As $M_f^*\psi\in L^2\Rightarrow f\psi\in L^2\Rightarrow\psi\in D(M_f)$.

b) „$\subseteq$" $\lambda\notin\operatorname{ess\,ran}f\Rightarrow (f-\lambda)^{-1}\in L^\infty(\mu)\Rightarrow M_{(f-\lambda)^{-1}}$ bounded on $L^2(\mu)$ and $M_{(f-\lambda)^{-1}}=(M_f-\lambda)^{-1}\Rightarrow\lambda\notin\operatorname{spec}(M_f)$.

„$\supseteq$" $\lambda\in\operatorname{ess\,ran}f$, $\varepsilon_m:=2^{-m}$, $X_m:=\{x: |f(x)-\lambda|<\varepsilon_m\}$

If $\mu(X_m) = \infty$, replace X_m by a subset of finite measure

$\Longrightarrow \|(M_f - \lambda) \mathbb{1}_{X_m}\| \leq 2^{-m} \|\mathbb{1}_{X_m}\| \Longrightarrow M_f - \lambda$ cannot have bounded inverse.

$\Longrightarrow \lambda \in \mathrm{Spec}(M_f)$ $\boxed{}$

In the following $\mathcal{H}$ will denote a separable Hilbert space.

Thm 2.5.1/3: $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ selfadjoint

$\Longrightarrow \exists\ \mu$ (finite) measure on $\mathrm{Spec}(H) \times \mathbb{N}$

$\exists\ U : \mathcal{H} \to L^2 := L^2(\mathrm{Spec}(H) \times \mathbb{N}, \mu)$ unitary s.t.

if $h : \mathrm{Spec}(H) \times \mathbb{N} \to \mathbb{R}$
$(s, n) \longmapsto s$

1.) $\xi \in D(H) \Longleftrightarrow U(\xi) \in D(M_h)$

2.) $U H U^{-1} \psi = M_h \psi \quad \forall \psi \in U(D(H))$

3.) $U f(H) U^{-1} \psi = M_{f \circ h} \psi \quad \forall f \in C_0(\mathbb{R})\ \forall \psi \in L^2$.

4.) $\mathrm{Spec}(H) = \mathrm{ess\,ran}(h)$.

The map in 3.) extends uniquely to a $*$-homomorphism

$$\left\{\begin{matrix}\text{bounded} \\ \text{Borel-meas.} \\ \text{fcts on } \mathbb{R}\end{matrix}\right\} \longrightarrow \mathbb{B}(\mathcal{H}) \qquad \text{s.t.} \quad f_n(H)\xi \xrightarrow{n\to\infty} f(H)\xi$$
$$f \longmapsto f(H) \qquad\qquad \forall f_n \nearrow f \quad \forall \xi \in L^2.$$

Proof: e.g. Davies. $\boxed{}$

The projections $\chi_\Omega(H) =: P_\Omega$ are called the spectral projections of H onto the measurable subset $\Omega \subset \mathbb{R}$. $L_\Omega := P_\Omega \mathcal{H}$ spectral subspaces

Note that $L_\Omega = U^{-1} L^2(E_\Omega, \mu)$, $E_\Omega = \{(s,n) \in \mathrm{Spec}(H) \times \mathbb{N},\ h(s,n) \in \Omega\}$.

$P_\Omega \neq 0 \Longleftrightarrow \Omega \cap \mathrm{ess\,ran}\, h \neq \emptyset$.

$\{P_\Omega\}_{\Omega \text{ measurable}}$ has the following properties:

a) $P_\Omega^2 = P_\Omega = P_\Omega^*$ (P_Ω is an orthogonal projection)

b) $P_\emptyset = 0$, $P_{\mathbb{R}} = \mathbb{1}$

c) If $\Omega = \bigcup_n \Omega_n$, $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$ $\Rightarrow$ $P_\Omega \xi = \lim_{N\to\infty} \sum_{n=1}^N P_{\Omega_n} \xi$ $\forall \xi \in \mathcal{H}$.

d) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

A family satisfying these properties is called a <u>projection-valued measure</u>.

For any $\xi \in \mathcal{H}$ it gives rise to a Borel measure on $\mathbb{R}$ $(\xi, P_\Omega \xi)$ denoted by $d(\xi, P_\lambda \xi)$. By polarization, we obtain a complex measure

$$d(\xi, P_\lambda \eta) := \frac{1}{4} d(\xi+\eta, P_\lambda(\xi+\eta)) - \frac{1}{4} d(\xi-\eta, P_\lambda(\xi-\eta))$$
$$+ \frac{i}{4} d(\xi+i\eta, P_\lambda(\xi+i\eta)) - \frac{i}{4} d(\xi-i\eta, P_\lambda(\xi-i\eta)), \quad \xi,\eta \in \mathcal{H}.$$

It is not difficult to show that $\forall \xi \in \mathcal{H}$:

$$(\xi, f(H)\xi) = \int_{-\infty}^{\infty} f(\lambda)\, d(\xi, P_\lambda \xi).$$

Suppose that f is a Borel fct., possibly unbounded, and let

$$D(f(H)) := \{\xi \in \mathcal{H} : \int_{-\infty}^{\infty} |f(\lambda)|^2 d(\xi, P_\lambda \xi) < \infty\}.$$

Then $D(f(H)) \subset \mathcal{H}$ dense and we may define $f(H) : D(f(H)) \subset \mathcal{H} \to \mathcal{H}$ from $(\xi, f(H)\xi) := \int_{-\infty}^{\infty} f(\lambda)\, d(\xi, P_\lambda \xi)$ $\forall \xi \in D(f(H))$.

Symbolically: $f(H) = \int_{-\infty}^{\infty} f(\lambda)\, dP_\lambda$. We have basically deduced:

<u>Thm:</u> (spectral decomposition)

There is a bijective correspondence between projection-valued measures $\{P_\Omega\}$ on $\mathcal{H}$ and self-adjoint operators A:

$$A = \int_{-\infty}^{\infty} \lambda\, dP_\lambda.$$

Example / Thm 3.5.3:

Let $a: \mathbb{R}^n \to \mathbb{R}$ measurable, $op(a) = \mathcal{F}^{-1}_{\xi \to x} a(\xi) \mathcal{F}_{x \to \xi}$,

e.g. $a(\xi) = \sum a_\alpha \xi^\alpha$, $op(a): D(a) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$,

$D(a) = \{ \varphi \in L^2(\mathbb{R}^n) : a(\xi)\hat{\varphi}(\xi) \in L^2(\mathbb{R}^n) \}$

$\Longrightarrow$ a) $op(a)$ unitarily equivalent to M_a by definition, Spectral thm with $U = \mathcal{F}^{-1}$! (Except that μ is not finite here).

b) $op(a)$ self-adjoint because M_a is so.

c) $\mathrm{Spec}(op(a)) = \mathrm{Spec}(M_a) = \mathrm{ess\,ran}\, a$

d) $f(op(a)) = op(f \circ a)$

Some measure theory:

Lemma 4.1.1: μ finite measure on $\mathbb{R}^N$, $E \subset \mathbb{R}^N$ Borel subset

$\Longrightarrow \dim L^2(E, \mu) = m < \infty \Longleftrightarrow \exists\, x_1, \ldots, x_m \in E$: $\cdot\ \mu(\{x_j\}) > 0\ \forall j$

$\cdot\ \mu(E \setminus \{x_1, \ldots, x_m\}) = 0$

Proof: „$\Longleftarrow$": $L^2(E, \mu) = \mathrm{span}\{ \chi_{\{x_j\}} : j = 1, \ldots, m \}$,

„$\Longrightarrow$": Let $r \in \mathbb{N}$, Partition $\mathbb{R}^N$ into disjoint cubes $\{C_{r,j}\}_{j=1}^{\infty}$ of edge length 2^{-r}, lower left corner in $2^{r}\mathbb{Z}^N$.

$\Rightarrow \dim \underbrace{\mathrm{span}\{\chi_{C_{r,j}}\}}_{\in L^2} \leq \dim L^2 = m \quad \forall r \in \mathbb{N}$

$\Rightarrow \bigcup_{j=1}^{\infty} \{ C_{r,j} : \mu(C_{r,j}) \neq 0 \} \xrightarrow{r \to \infty} \{x_1, \ldots, x_r\}$, $r \leq m$.

By „$\Longleftarrow$", $r = m$

$\boxtimes$

Lemma 4.1.2: $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ selfadjoint. Then

$$\lambda \in \operatorname{Spec}(H) \iff \exists\ (f_n)_{n=1}^{\infty} \subseteq D(H) : \ \|f_n\|_{\mathcal{H}} = 1 \text{ and } \lim_{n\to\infty} \|Hf_n - \lambda f_n\|_{\mathcal{H}} = 0 .$$

Proof: If such a sequence exists $\Rightarrow$ $H-\lambda$ cannot be boundedly invertible.
$\Rightarrow \lambda \in \operatorname{Spec}(H)$

If $\lambda \in \operatorname{Spec}(H) \Rightarrow \lambda \in \operatorname{ess\,ran} h$. Let $f_n := U^{-1} 1\!\!1_{X_n}$,

$X_n \subseteq \{(s,n) : |h(s,n) - \lambda| < 2^{-n}\}$ of finite measure > 0. $\boxed{}$

λ eigenvalue of H $\iff$ $E := \{(\lambda, n) : n \in \mathbb{N}\}$ has measure > 0

$\iff$ $U^{-1} L^2(E, \mu)$ eigenspace to λ

$\sigma_p(H)$ = point spectrum of H $= \bigcup \{$eigenvalues of $H\}$

$\sigma_{disc}(H)$ = discrete spectrum of $H = \bigcup \{$ eigenvalues of H s.t. $\exists \varepsilon > 0$: $(\lambda - \varepsilon, \lambda + \varepsilon) \cap \operatorname{Spec}(H) = \lambda$ and λ has finite multiplicity$\}$

$\sigma_{ess}(H) = \operatorname{Ess\,Spec}(H) = \operatorname{Spec}(H) \setminus \sigma_{disc}(H)$.

Example: For a Fourier multiplier, $\operatorname{Ess\,Spec}(H) = \operatorname{Spec}(H)$

$=$ ess ran (symbol). A similar result holds for globally estimated pseudo differential operators on $\mathbb{R}^n$, see Gerd Grubb, Remark 7.16.

Lemma 4.1.4: • $\operatorname{Ess\,Spec}(H) \subset \operatorname{Spec}(H)$ closed

• $\lambda \in \operatorname{EssSpec}(H) \iff \dim P_{(\lambda-\varepsilon, \lambda+\varepsilon)} \mathcal{H} = \infty \ \forall \varepsilon > 0$.

Proof: • Discrete spectrum = isolated points = relatively open

$\Rightarrow$ Ess Spec $= \operatorname{Spec}(H) \setminus \{$discrete spectrum$\}$ closed

• "$\Rightarrow$" only: $\dim P_{(\lambda-\varepsilon,\lambda+\varepsilon)}\mathcal{H} = \dim L^2(E_{(\lambda-\varepsilon,\lambda+\varepsilon)}, \mu) < \infty$ for some $\varepsilon > 0$

$\Rightarrow \exists (s_1, n_1), \dots, (s_m, n_m) \in E_{(\lambda-\varepsilon,\lambda+\varepsilon)}$ whose complement has measure 0
s_r eigenvalues of finite multiplicity, $(\lambda-\varepsilon, \lambda+\varepsilon) \setminus \{s_1, \dots, s_m\} \cap \operatorname{Spec}(H) = \emptyset$

$\Rightarrow \exists r : \lambda = s_r$ $\boxed{}$

Thm: Let $H: D(H) \subset \mathcal{H} \to \mathcal{H}$ selfadjoint, $\dim \mathcal{H} = \infty$.

a) (4.1.5, 4.2.3) The following are equivalent:

① $\operatorname{Ess\,Spec} H = \emptyset$

② $\exists \lambda \notin \operatorname{Spec}(H): (H-\lambda)^{-1}$ compact

③ $\forall \lambda \notin \operatorname{Spec}(H): (H-\lambda)^{-1}$ compact

④ $\exists$ ONB $\{e_n\}_{n=1}^{\infty} \subseteq D(H)$ s.t. $He_n = \lambda_n e_n$, $|\lambda_n| \nearrow \infty$.

b) (8.4.1). The following are equivalent:

① $\lambda \in \operatorname{Ess\,Spec}(H)$

② $\forall \varepsilon > 0 \ \forall n \in \mathbb{N} \ \exists$ subspace $L_{n,\varepsilon} \subseteq D(H)$ s.t. $\dim L_{n,\varepsilon} \geq n$ and $\forall f \in L_{n,\varepsilon}: \|Hf - \lambda f\| \leq \varepsilon \|f\|$

③ $\forall \varepsilon > 0 \ \exists$ subspace $L_\varepsilon \subseteq D(H)$ s.t. $\dim L_\varepsilon = \infty$ and $\forall f \in L_\varepsilon: \|Hf - \lambda f\| \leq \varepsilon \|f\|$

④ $H - \lambda: D(H) \to \mathcal{H}$ is not Fredholm

c) (Weyl) If $K: D(H) \to \mathcal{H}$ compact $\Rightarrow \operatorname{EssSpec}(H) = \operatorname{EssSpec}(H+K)$

Here $D(H)$ is endowed with the graph norm, $\|h\|_{D(H)} := \|h\| + \|Hh\|$

More precisely: $\operatorname{Ess\,Spec}(H) = \bigcap_{K \text{ compact}} \operatorname{Spec}(H+K)$.

d) (8.4.3) Let $\tilde{H}: D(\tilde{H}) \subset \mathcal{H} \to \mathcal{H}$ selfadjoint s.t.

$\exists \lambda_0 \notin \operatorname{Spec}(H) \cup \operatorname{Spec}(\tilde{H})$: $(H-\lambda_0)^{-1} - (\tilde{H}-\lambda_0)^{-1}$ compact

$\Longrightarrow \operatorname{EssSpec}(H) = \operatorname{Ess\,Spec}(\tilde{H})$.

Proof: a) ① ⇒ ④: Ess Spec $H = \emptyset \Rightarrow$ Spec $H = \bigcup_{r=1}^{\infty} \{s_r\}$ union of isolated eigenvalues with $\pm\infty$ as only accumulation points and of finite multiplicity. Wlog: $|s_1| \leq |s_2| \leq \ldots$, ortho. eigenvectors $\{f_n\}$.

Assume $L' := \mathrm{span}\{f_n : n \in \mathbb{N}\}$ is not dense in $\mathcal{H}$.

$\Longrightarrow$ $L := L'^{\perp} \neq \{0\}$ is invariant under H in the sense that $(z-H)^{-1} L \subseteq L$ $\forall z \notin \mathbb{R}$: If $\psi \in L$ and $\phi \in L' \Rightarrow \langle (z-H)^{-1}\psi, \phi\rangle = \langle \psi, \underbrace{(\bar{z}-H)^{-1}\phi}_{= \sum_{r=1}^{\infty} \frac{c_r}{\bar{z}-s_r} f_r \in L', \text{ if } \phi = \sum c_r f_r} \rangle \overset{L = L'^{\perp}}{=} 0$, so

$(z-H)^{-1}\psi \in L'^{\perp} = L$.

L is closed (as an orthogonal complement) subspace of $\mathcal{H}$, so it is a Hilbert space and $H|_L : D(H) \cap L \to L$ is selfadjoint $\Rightarrow$ Spec$(H|_L) \neq \emptyset \Rightarrow \exists$ further eigenvalues $\lightning$.

$\{f_n\}$ basis.

④ ⇒ ①: By assumption, Spec$(H) = \bigcup_{n=1}^{\infty} \{\lambda_n\} \Rightarrow$ EssSpec$(H) = \emptyset$.

② ⇒ ③: (See Chapter 1) If $(\lambda_0 - H)^{-1}$ compact $\Rightarrow \forall \lambda \notin$ Spec(H):

$$\lambda - H = (\lambda_0 - H)\left(\mathbb{1} + (\lambda - \lambda_0)(\lambda_0 - H)^{-1}\right), \text{ so}$$

$$(*) \quad (\lambda - H)^{-1} = \underbrace{\left(\mathbb{1} + (\lambda-\lambda_0)(\lambda_0 - H)^{-1}\right)^{-1}}_{\text{bounded}} \underbrace{(\lambda_0 - H)^{-1}}_{\text{compact}} \quad \text{compact.}$$

③ ⇒ ④: (See chapter 1.) $(H-\lambda_0)^{-1}$ compact for some $\lambda_0 \notin$ spec(H) $\overset{(H-\lambda_0)^{-1} \text{ normal}}{\Longrightarrow}$ Spec$((H-\lambda_0)^{-1}) = \bigcup_{j=1}^{\infty} \{r_j\}$ eigenvalues. As $(H-\lambda_0)^{-1}$ injective $\Rightarrow r_j \neq 0$ and

$$(H-\lambda_0)^{-1} e_j = r_j e_j \iff H e_j = \underbrace{(\lambda_0 + r_j^{-1})}_{=: \lambda_j} e_j \qquad (**)$$

$\Rightarrow$ Spec $H \supseteq \bigcup_{j=1}^{\infty} \{\lambda_j\}$. But $(*)$ shows that $\lambda \notin$ Spec H

$\Rightarrow$ $\mathbb{1} + (\lambda - \lambda_0)(\lambda_0 - H)^{-1}$ invertible $\Rightarrow \frac{1}{\lambda - \lambda_0} \notin$ Spec$(\lambda_0 - H)$

$\Rightarrow$ spec $H = \bigcup_{j=1}^{\infty} \{\lambda_j\}$ countable.

Let $\lambda_0 \in \mathbb{R} \setminus$ Spec $H \Rightarrow (\lambda_0 - H)^{-1}$ selfadjoint

$\Rightarrow \exists$ ONB $\{e_n\} \subseteq D(H)$ s.t. $(\lambda_0 - H)^{-1} e_j = r_j e_j$, $r_j \neq 0$

By $(**)$, the e_i are an ONB of eigenvectors of H.

④ ⇒ ②: Wlog: $|\lambda_1| \le |\lambda_2| \le \dots \to \infty$, $-1 \ne \lambda_j \ \forall j$. Let $A_n \in \mathbb{K}(\mathcal{H})$

$$A_n f := \sum_{r=1}^{n} (\lambda_r + 1)^{-1} \langle f, e_r \rangle e_r \quad \text{, so that}$$

$$\{(H+1)^{-1} - A_n\} f = \sum_{r=n+1}^{\infty} (\lambda_r + 1)^{-1} \langle f, e_r \rangle e_r \quad \text{or}$$

$$\| \{(H+1)^{-1} - A_n\} f \| \le (\lambda_n + 1)^{-1} \|f\| \xrightarrow{n \to \infty} 0.$$

So $(H+1)^{-1}$ belongs to the norm closure of compact operators $\Rightarrow (H+1)^{-1}$ compact.

b) ① ⇒ ③: Set $L_\varepsilon := L_{(\lambda-\varepsilon, \lambda+\varepsilon)}$ in Lemma 4.1.4.

③ ⇒ ②: Take any n-dim. subspace of L_ε.

② ⇒ ①: Suppose $\dim L_{(\lambda-\varepsilon,\lambda+\varepsilon)} < n < \infty$ for some $\varepsilon > 0$,

$P :=$ orth. projection from $\mathcal{H}$ onto $L_{(\lambda-\varepsilon,\lambda+\varepsilon)}$.

As $n > \dim L_{(\lambda-\varepsilon,\lambda+\varepsilon)}$, $P|_{L_{n,\varepsilon/2}} : L_{n,\varepsilon/2} \to L_{(\lambda-\varepsilon,\lambda+\varepsilon)}$

is not injective $\Rightarrow \exists\, 0 \ne f \in L_{n,\varepsilon/2} \cap L_{(\lambda-\varepsilon,\lambda+\varepsilon)}^{\perp}$

Spectral thm: $\|Hf - \lambda f\|_{\mathcal{H}} = \| M_h U f - \lambda U f \|_{L^2}$

$$= \| \underbrace{(h-\lambda)}_{\ge \varepsilon, \text{ as } f \in L_{(\lambda-\varepsilon,\lambda+\varepsilon)}^{\perp}} U f \|_{L^2} \ge \varepsilon \|Uf\| = \varepsilon \|f\| \ \text{↯}$$

$\Rightarrow \dim L_{(\lambda-\varepsilon,\lambda+\varepsilon)} = \infty \Rightarrow \lambda \in \mathrm{Ess\,Spec}(H)$.

③ ⇒ ④: Let $H-\lambda$ Fredholm $\Rightarrow \mathrm{ran}(H-\lambda)$ closed.

Note $\tilde{H}_\lambda : D(H-\lambda)/_{\ker(H-\lambda)} \longrightarrow \mathrm{ran}(H-\lambda)$ bijective

$$[f] \longmapsto (H-\lambda) f$$

Closed graph thm $\Rightarrow$: $\tilde{H}_\lambda^{-1}$ bounded

$\Rightarrow \exists \varepsilon > 0 \ \|(H-\lambda) f\| \ge \varepsilon \|f\| \quad \forall f \perp \ker(H-\lambda) \Rightarrow$ negation of ③

④ ⇒ ③: If $H-\lambda$ not Fredholm $\Rightarrow \dim \ker(H-\lambda) = \infty$

(note H selfadjoint!) Choose $L_\varepsilon := \ker H - \lambda$

c) $\lambda \notin \mathrm{Ess\,Spec}(H) \overset{b)④}{\iff} H-\lambda$ Fredholm $\iff H+K-\lambda$ Fredholm

$\iff \lambda \notin \mathrm{Ess\,Spec}(H+K)$

To see $\bigcap_{K \text{ compact}} \mathrm{Spec}(H+K) \supset \mathrm{Ess\,Spec}(H)$, let $\lambda \notin \bigcap \mathrm{Spec}(H+K) \Rightarrow \exists K$: $H+K-\lambda$ invertible

$\Rightarrow H - \lambda = H+K-\lambda-K$ Fredholm $\Rightarrow \lambda \notin \mathrm{Ess\,Spec}(H)$. Conversely, if $\lambda \notin \mathrm{Ess\,Spec}(H)$ either $\lambda \notin \mathrm{Spec}(H)$ or $\lambda \in \sigma_{disc}(H)$. Then $H + P_\lambda - \lambda$ invertible, so $\lambda \notin \bigcap \mathrm{spec}(H+K)$.

d) Let $\lambda_0 \notin \mathrm{Spec}(H)$. $\Rightarrow$ (*) Shows that

$\lambda \in \mathrm{Ess\,Spec}(H) \Leftrightarrow \frac{1}{\lambda - \lambda_0} \in \mathrm{Ess\,Spec}((\lambda_0 - H)^{-1})$. Shorter: $\overset{e)}{\Leftrightarrow} \frac{1}{\lambda - \lambda_0} \in \mathrm{EssSpec}((\lambda_0 - \tilde{H})^{-1}) \Leftrightarrow \lambda \in \mathrm{EssSpec}(\tilde{H})$

If $\lambda \in \mathrm{Ess\,Spec}(H)$, $\varepsilon > 0$ $\overset{b)④}{\Longrightarrow}$ $\exists$ inf-dim. subspace $L \subset \mathcal{H}$ $\forall f \in L$:

$$\|(H-\lambda_0)^{-1} f - (\lambda - \lambda_0)^{-1} f\| \leq \varepsilon \|f\|$$

As $(H-\lambda_0)^{-1}$ bounded $\Rightarrow$ May assume L closed. Assume $\lambda_0 \notin \mathrm{Spec}(H) \cup \mathrm{Spec}(\tilde{H})$)

Let $B := P((H-\lambda_0)^{-1} - (\tilde{H}-\lambda_0)^{-1})^{*}((H-\lambda_0)^{-1} - (\tilde{H}-\lambda_0)^{-1}) P|_L$, P being the orth. projection from $\mathcal{H}$ onto L. $\Rightarrow$ B compact on L, so by b) ④ $0 \in \mathrm{Ess\,Spec}(B)$

$\overset{b)⑤}{\Longrightarrow}$ $\exists$ inf. dim. subspace M of L s.t. $\|Bf\| \leq \varepsilon^2 \|f\|$ $\forall f \in M$

$$\Rightarrow \forall f \in M \subset L: \underbrace{\|((H-\lambda_0)^{-1} - (\tilde{H}-\lambda_0)^{-1}) P f\|^2}_{= (Bf,\, f)} \leq \|Bf\|\,\|f\| \leq \varepsilon^2 \|f\|^2$$

$$\|((H-\lambda_0)^{-1} - (\tilde{H}-\lambda_0)^{-1}) f\|^2 =$$

$$\Rightarrow \forall f \in M \subset L \quad \|(\tilde{H}-\lambda_0)^{-1} f - (\lambda-\lambda_0)^{-1} f\|$$

$$\leq \|(\tilde{H}-\lambda_0)^{-1} f - (H-\lambda_0)^{-1} f\| + \|(H-\lambda_0)^{-1} f - (\lambda-\lambda_0)^{-1} f\|$$

$$\leq \varepsilon\|f\| + \varepsilon\|f\| = 2\varepsilon\|f\|$$

$\Rightarrow \frac{1}{\lambda - \lambda_0} \in \mathrm{EssSpec}((\lambda_0 - \tilde{H})^{-1}) \Rightarrow \lambda \in \mathrm{EssSpec}(\tilde{H})$.

Conclusion: $\mathrm{EssSpec}(H) \subseteq \mathrm{EssSpec}(\tilde{H})$. Exchanging the roles of $H, \tilde{H}$

$\Rightarrow \mathrm{Ess\,Spec}(H) = \mathrm{Ess\,Spec}(\tilde{H})$ ☐

A typical application of d) is to show that $\mathrm{EssSpec}(H) = [0, \infty)$ for the operator $H = -\Delta + \frac{1}{|x|} : H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$. Indeed, let $H_0 := -\Delta : H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R})$. From Kim's talk we know that H and H_0 are self-adjoint and bounded from below, and we are going to show that $(H_0 - \lambda)^{-1} - (H-\lambda)^{-1}$ is compact.

The operator $f(x)\,g(D)$:

Theorem: Let $f, g \in L^p(\mathbb{R}^N)$, $2 \le p < \infty$, or $f, g \in L^\infty_0$, meaning $f, g \in L^\infty$ and $\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} g(x) = 0$. Then the operator defined by

$f(x)\,g(D)\,u(x) := f(x)\,\mathcal{F}^{-1}_{\xi\to x}(g(\xi)\,\hat{u}(\xi))$ is compact on $L^2(\mathbb{R}^N)$.

Moreover, $\|f(x)\,g(D)\| \le C_{N,p}\,\|f\|_p\,\|g\|_p$ $(*)$

Remark: Note that $f(x)\,g(D)$ is the composition of two operators, which are not necessarily bounded.

Proof: Let $\varphi \in L^2$, $\psi = \mathcal{F}(g(D)\varphi) = g\,\mathcal{F}\varphi$. Hölder's inequality:

$\|\psi\|_{L^r} \le \|g\|_{L^p}\,\|\mathcal{F}\varphi\|_{L^2} = \|g\|_{L^p}\,\|\varphi\|_{L^2}$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{2}$, so $\psi \in L^r$ for some $1 < r \le 2$.

$\Rightarrow$ (Hausdorff-Young inequality, DifFun2, Ch1: $\mathcal{F}^{-1}: L^r \xrightarrow{\text{bdd}} L^{r'}$, $\frac{1}{r} + \frac{1}{r'} = 1$)

$g(D)\varphi = \mathcal{F}^{-1}\psi \in L^{r'}$, $\frac{1}{r'} = 1 - \frac{1}{r} = \frac{1}{2} - \frac{1}{p}$, so $r' \in [2, \infty)$.

$$\overset{\text{Hölder}}{\Longrightarrow} \|\underbrace{f(x)}_{\in L^p}\,\underbrace{g(D)\varphi}_{\in L^{r'}}\|_{L^2} \le \|f\|_{L^p}\,\|g(D)\varphi\|_{L^{r'}} \le \|\mathcal{F}^{-1}\|\,\|f\|_{L^p}\,\|g\|_{L^p}\,\|\varphi\|_{L^2}$$

$\Longrightarrow$ $f(x)\,g(D)$ bounded and $(*)$ holds

As L^∞_{comp} is dense in L^p, $p < \infty$, and in L^∞_0:

$\exists\, f_n, g_n \in L^\infty_{comp}$: $\|f_n - f\|_{L^p} \to 0$, $\|g_n - g\|_{L^p} \to 0$. $(*) \Longrightarrow$

$$\|f_n(x)\,g_n(D) - f(x)\,g(D)\| = \|(f_n(x) - f(x))\,g_n(D) - f(x)(g(D) - g_n(D))\|$$

$$\overset{(*)}{\le} C\,\underbrace{\|f_n - f\|_p}_{\to 0}\,\|g_n\|_p + C\,\|f\|_p\,\underbrace{\|g - g_n\|_p}_{\to 0} \quad (**)$$

and the integral kernel of $f_n(x)\,g_n(D)$ is $f_n(x)\,(\mathcal{F}^{-1}g_n)(x-y)$.

Note that $\int |\text{Kernel}|^2 = \int |f_n(x)|^2\,|(\mathcal{F}^{-1}g_n)(x-y)|^2\,dx\,dy \underset{\text{Plancherel}}{=} \|f_n\|^2_{L^2}\,\|g_n\|^2_{L^2} < \infty$

is Hilbert-Schmidt, in particular compact $\Rightarrow$ By $(**)$ $f(x)\,g(D)$ compact $\boxed{}$

Cor: $f \in L^p(\mathbb{R}^N) + L^\infty_0(\mathbb{R}^N)$, $g \in L^p(\mathbb{R}^N) \cap L^\infty_0(\mathbb{R}^N)$

$\Longrightarrow$ $f(x)\, g(D)$ compact on L^2.

Proof: Write $f = f_p + f_\infty$, $f_p \in L^p$, $f_\infty \in L^\infty_0$.

$$\Longrightarrow \quad f(x)\, g(D) = \underbrace{f_p(x)\, g(D)}_{\text{compact}} + \underbrace{f_\infty(x)\, g(D)}_{\text{compact}} \quad \text{compact.}$$

Cor: Let $V \in L^2(\mathbb{R}^3) + L^\infty_0(\mathbb{R}^3)$. Then $\mathrm{Ess\,Spec}(-\Delta + V) = [0,\infty)$.

Proof: As noted above, $H := -\Delta + V$, $H_0 := -\Delta$: $H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ are selfadjoint, bounded from below and, by Example/Thm 3.5.3 (the spectrum of Fourier multipliers), $\mathrm{Ess\,Spec}(H_0) = [0,\infty)$. Note that

$$(H_0+\lambda)^{-1} - (H+\lambda)^{-1} = (H+\lambda)^{-1}(H - H_0)(H_0+\lambda)^{-1} = (H+\lambda)^{-1} V (-\Delta+\lambda)^{-1}$$

$(H+\lambda)^{-1} : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3) = D(H) \subset L^2$ is bounded for large λ, because H is bounded from below. As $(-\Delta+\lambda)^{-1} = g_\lambda(D)$, $g_\lambda(\xi) = \frac{1}{|\xi|^2+\lambda}$ $\in L^2(\mathbb{R}^3) \cap L^\infty_0(\mathbb{R}^3)$ for $\lambda > 0$, the above Corollary shows that $V(x)\, g_\lambda(D) = V(-\Delta+\lambda)^{-1}$ is compact. Therefore

$$(H_0+\lambda)^{-1} - (H+\lambda)^{-1} = \underbrace{(H+\lambda)^{-1}}_{\text{bounded}} \underbrace{\left(V(-\Delta+\lambda)^{-1}\right)}_{\text{compact}}$$

is compact. Part d) of our theorem about Ess Spec now says that $\mathrm{EssSpec}(H) = \mathrm{EssSpec}(H_0)$. $\boxtimes$

As in Kim's talk, the discussion extends to many-body Schrödinger operators. Davies has some more general examples in Chapter 8.

Further remarks on the spectral theorem

1.) Resolvents and spectral projections:

Let $f_\varepsilon(x) := \frac{1}{2\pi i} \int_a^b \left(\frac{1}{x-\lambda-i\varepsilon} - \frac{1}{x-\lambda+i\varepsilon} \right) d\lambda = \frac{\varepsilon}{\pi} \int_a^b \frac{d\lambda}{(x-\lambda)^2+\varepsilon^2}$

$= -\frac{1}{\pi}\left[\arctan\left(\frac{x-\lambda}{\varepsilon} \right) \right]_{\lambda=a}^{b}.$

Note $f_\varepsilon(x) \to \begin{cases} 0 & , x \notin [a,b] \\ 1/2 & , x \in \{a,b\} \\ 1 & , x \in (a,b) \end{cases} = \frac{1}{2}\left(\chi_{[a,b]}(x) + \chi_{(a,b)}(x) \right)$ monotonously.

Therefore the monotone convergence property of the functional calculus gives:

Thm (Stone) Let $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ selfadjoint $\Rightarrow \forall f \in \mathcal{H}$:

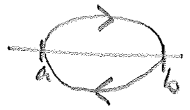
$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \{ (H-\lambda-i\varepsilon)^{-1} - (H-\lambda+i\varepsilon)^{-1} \} f \, d\lambda = \frac{1}{2} \{ P_{[a,b]} f + P_{(a,b)} f \}.$$

Interpreting the left hand side as

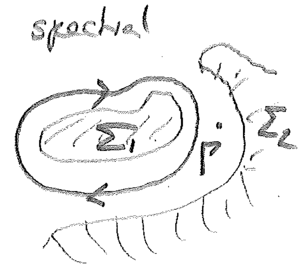
$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \left\{ \int (H-\lambda)^{-1} f + \int (H-\lambda)^{-1} f + \underbrace{\int (H-\lambda)^{-1} f}_{\xrightarrow[\varepsilon \to 0^+]{} 0} \right\} = \frac{1}{2\pi i} \int (H-\lambda)^{-1} f$$

if $a,b \notin \mathrm{Spec}(H)$, we obtain

Cor: If $a,b \notin \mathrm{Spec}(H)$, $P_{[a,b]} = P_{(a,b)} = \frac{1}{2\pi i} \int_\Gamma (H-\lambda)^{-1} d\lambda$,

where $\Gamma =$ 

.

Remark: Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ not necessarily selfadjoint and $\mathrm{Spec}(H) = \Sigma_1 \cup \Sigma_2$ a decomposition of $\mathrm{Spec}(H)$ into open subsets. If Σ_1 is bounded, one can <u>define</u> a spectral projection P_{Σ_1} as $P_{\Sigma_1} = \frac{1}{2\pi i} \int_\Gamma (H-\lambda)^{-1} d\lambda$, where 

2. Measure-theoretic decomposition of the spectrum

Apart from the decomposition $\mathrm{Spec}(H) = \sigma_p(H) \cup \sigma_{ess}(H) = \sigma_{disc}(H) \cup \sigma_{ess}(H)$

sometimes a measure-theoretic decomposition is used.

Let $\mathfrak{A}$ be a σ-algebra, μ, ν σ-finite measures on $\mathfrak{A} \subseteq \mathcal{P}(X)$.

Def: a) μ is absolutely continuous wrt ν if $\nu(A) = 0 \Rightarrow \mu(A) = 0$. $\forall A \in \mathfrak{A}$.

b) μ and ν are mutually singular if $\exists A \in \mathfrak{A} : \mu(A) = \nu(X \setminus A) = 0$.

The two basic theorems we are going to need are the following:

Thm: (Radon-Nikodym)

μ absolutely continuous wrt ν $\Longleftrightarrow$ $\exists f : X \to [0, \infty]$ measurable s.t. $\forall A \in \mathfrak{A} : \mu(A) = \int_A f(x)\, d\nu(x)$.

Thm: (Lebesgue decomposition)

There exists a unique decomposition $\mu = \mu_{ac} + \mu_{sing}$, where ν and μ_{sing} are mutually singular and μ_{ac} is absolutely cont. wrt ν.

In particular, using the set A from the definition of "mutually singular",

$$L^2(X, \mu) = L^2(A, \mu_{sing}) \oplus L^2(X \setminus A, f\nu) = L^2(X, \mu_{sing}) \oplus L^2(X, \mu_{ac})$$

$$\begin{array}{ccc} \psi & & \\ g & \longmapsto & (g|_A, g|_{X\setminus A}) \end{array}$$

as $\int_X |g|^2\, d\mu = \int_X |g|^2\, d\mu_{ac} + \int_X |g|^2\, d\mu_{sing} = \int_{X\setminus A} |g|^2\, d\mu_{ac} + \int_A |g|^2\, d\mu_{sing}$

We now restrict to Borel measures on $S := \mathrm{spec}(A)$

Def: a) μ is continuous if $\mu(\{x\}) = 0 \quad \forall x \in S$

b) μ is a pure point measure if $\mu(X) = \sum_{x \in X} \mu(\{x\}) \quad \forall$ Borel X.

Prop: There exists a unique decomposition $\mu = \mu_{pp} + \mu_{cont}$ s.t. μ_{cont} is continuous and μ_{pp} is a pure point measure.

Proof: Uniqueness is clear. To show existence, note that $\mu(C) < \infty$ $\forall C$ compact so that $\{x \in \mathbb{R} : \mu(\{x\}) > 0\}$ is countable. The measures $\mu_{pp}(X) := \sum_{x \in X} \mu(\{x\}) \geq 0$ and $\mu_{cont} = \mu - \mu_{pp} \geq 0$ give the desired decomposition. $\square$

Letting $A :=$ support of μ_{pp} we see that μ_{pp} and μ_{cont} are mutually singular.

Cor: (Lebesgue decomposition, refined version)

$$\mu = \mu_{ac} + \mu_{sing} + \mu_{pp} \quad \text{where}$$

- μ_{ac} is absolutely continuous wrt the Lebesgue measure
- μ_{sing} is continuous and singular wrt the Lebesgue measure
- μ_{pp} is pure point.

Furthermore, $L^2(S,\mu) = L^2(S,\mu_{ac}) \oplus L^2(S,\mu_{sing}) \oplus L^2(S,\mu_{pp})$.

Proof:

$$\mu = \mu_{ac} + \mu'_{sing} \qquad \text{(Lebesgue decomposition)}$$
$$= \mu_{ac} + \underbrace{(\mu'_{sing})_{cont}}_{\mu_{sing}} + \underbrace{(\mu'_{sing})_{pp}}_{\mu_{pp}} \qquad \text{(Proposition)}$$

The decomposition of L^2 follows as above from the mutual singularity $\square$

Exercise: Let H be the multiplication operator associated to $f(x) = x$ on $L^2(\mathbb{R},\mu)$, and let $\varphi \in L^2(\mathbb{R},\mu)$. Show that

- $\Omega \mapsto (\varphi, P_\Omega \varphi)_{L^2}$ is absolutely continuous $\overset{\text{wrt }\lambda}{\iff} \varphi \in L^2(\mathbb{R},\mu_{ac})$
- $\Omega \mapsto (\varphi, P_\Omega \varphi)_{L^2}$ is pure point $\iff \varphi \in L^2(\mathbb{R},\mu_{pp})$
- $\Omega \mapsto (\varphi, P_\Omega \varphi)_{L^2}$ is continuous, singular wrt $\lambda \iff \varphi \in L^2(\mathbb{R},\mu_{sing})$

In general, we can use the spectral theorem to define:

Def: Let $H: D(H) \subset \mathcal{H} \to \mathcal{H}$ selfadjoint. Define the H-invariant subspaces

a) $\mathcal{H}_{pp} := \{ f \in \mathcal{H} : \Omega \mapsto (f, P_\Omega f)_{\mathcal{H}}$ pure point $\}$

b) $\mathcal{H}_{ac} := \{ f \in \mathcal{H} : \Omega \mapsto (f, P_\Omega f)_{\mathcal{H}}$ abs. cont. wrt Lebesgue meas.$\}$

c) $\mathcal{H}_{sing} := \{ f \in \mathcal{H} : \Omega \mapsto (f, P_\Omega f)_{\mathcal{H}}$ cont. and singular wrt Lebesgue measure $\}$.

Def:

$\sigma_{pp}(H) := \{$ eigenvalues of $H \}$ — pure point spectrum (not necessarily closed)

$\sigma_{cont}(H) := \mathrm{Spec}(H|_{\mathcal{H}_{sing} \oplus \mathcal{H}_{ac}})$ — continuous spectrum (closed)

$\sigma_{ac}(H) := \mathrm{Spec}(H|_{\mathcal{H}_{ac}})$ — absolutely cont. spectrum (closed)

$\sigma_{sing}(H) := \mathrm{Spec}(H|_{\mathcal{H}_{sing}})$ — (continuous) singular spectrum (closed)

As $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$, $\mathrm{Spec}(H) = \overline{\sigma_{pp}} \cup \sigma_{ac} \cup \sigma_{sing}$,

but one cannot omit the closure of σ_{pp}:

If K is compact and $0 \in \mathrm{Spec}(K)$ is not an eigenvalue

$\Rightarrow$ $\mathrm{Spec}(K) = \{0\} \cup \sigma_{pp} = \overline{\sigma_{pp}}$. $\sigma_{sing}(K) = \sigma_{ac}(K) = \emptyset$.

One always has $\sigma_{cont} = \sigma_{ac} \cup \sigma_{sing}$ by definition.

Remark: The above definition is equivalent to saying that

$\lambda \in \sigma_X(H) \iff \exists n \in \mathbb{N} : (\lambda, n) \in \mathrm{supp}\, \mu_X$, $X \in \{pp, ac, sing, cont\}$

where μ is the measure from the multiplication operator version of the spectral theorem.

Variational principles

We consider a lower-bounded self-adjoint operator $H: D(H) \subset \mathcal{H} \to \mathcal{H}$. Wlog $H \geq C\mathbb{1}$. For any finite-dimensional subspace $L \subseteq D(H)$ denote

$$\lambda(L) := \sup \{ \langle Hf, f \rangle : f \in L, \|f\| = 1 \}$$

Define $\lambda_n := \lambda_n(H) := \inf \{ \lambda(L) : L \subseteq D(H), \dim L = n \}$.

Thm 4.5.1/2: a) If H has compact resolvent $\Rightarrow$ λ_n is the n-th eigenvalue of H and $\lambda_n \to \infty$ (if $\dim \mathcal{H} = \infty$)

b) Let $a := \inf \text{Ess Spec } H > -\infty$. Then either

- $\lambda_n \nearrow a$ as $n \to \infty$ and $\text{Spec}(H) \cap [C, a) = \bigcup_{j=1}^{\infty} \{\lambda_j\}$

or - $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N < a$, $\lambda_n = a \ \forall n \geq N$ and $\text{Spec}(H) \cap [C, a) = \{\lambda_1, \dots, \lambda_N\}$.

Proof: a) H has compact resolvent $\Rightarrow \exists$ ONB $\{f_n\}$ of eigenvectors, eigenvalues μ_n. Let $M_n := \text{span}\{f_r : 1 \leq r \leq n\}$.

$$\Rightarrow \forall f \in M_n: \langle Hf, f \rangle = \sum_{r=1}^{n} \mu_r |\langle f, f_r \rangle|^2 \leq \mu_n \sum_{r=1}^{n} |\langle f, f_r \rangle|^2 = \mu_n \|f\|^2 \Rightarrow \lambda_n \leq \mu_n.$$

If $L \subset D(H)$ is any n-dim subspace and $P: \mathcal{H} \to M_{n-1}$ orth. proj.

$\Rightarrow \dim P(L) < \dim L \Rightarrow \exists f \in L: Pf = 0$.

Since $0 = Pf = \sum_{r=1}^{n-1} \langle f, f_r \rangle f_r \Rightarrow \langle f, f_r \rangle = 0 \ \forall r \leq n-1$

$$\Rightarrow \langle Hf, f \rangle = \sum_{r=n}^{\infty} \mu_r |\langle f, f_r \rangle|^2 \geq \mu_n \sum_{r=n}^{\infty} |\langle f, f_r \rangle|^2 \geq \mu_n \|f\|^2$$

$\Rightarrow \lambda(L) \geq \mu_n$ for all such $L \Rightarrow \lambda_n \geq \mu_n$. Thus $\lambda_n = \mu_n$.

b) $\text{Spec}(H) \cap [C, a) = \bigcup_m \{\mu_m\}$, μ_m eigenvalues of finite multiplicity

If $\mu_m \xrightarrow{m \to \infty} b \Rightarrow b \in \text{Ess Spec}(H) \Rightarrow b = a$.

As above $\lambda_n \leq \mu_n$. In the proof of $\mu_n \leq \lambda_n$, $Pf = 0$

now implies (using the spectral theorem) $\operatorname{supp} Uf \subseteq \{(s,m) : h(s,m) \geq \mu_n\}$

$\Rightarrow \langle Hf, f\rangle = \langle M_h Uf, Uf\rangle \geq \mu_n \langle Uf, Uf\rangle = \mu_n \|f\|^2.$

If only finitely many eigenvalues $\mu_m \leq a$ exist $\Rightarrow \lambda_m = \mu_m$ for the eigenvalues as above and $\lambda_m \geq a$ for $m > N$.

Assume $a \in \operatorname{Ess\,Spec}(H)$ $\forall \varepsilon > 0$ $\dim \underbrace{L_{(a-\varepsilon, a+\varepsilon)}}_{=:L} = \infty$. Let $\{f_r\}_{r=1}^{\infty} \subseteq L$ orthonormal. Let $L_n := \operatorname{span}\{f_1, \dots, f_n\} \Rightarrow \lambda(L_n) \leq b + \varepsilon$ $\forall \varepsilon > 0$ $\forall n \in \mathbb{N}$.

$\Rightarrow \lambda_n = a$ $\forall n > N$. $\boxed{4}$

Instead of considering n-dim subspaces of $D(H)$, it is often more convenient to work with the associated quadratic form $Q(f) := \langle H^{1/2} f, H^{1/2} f\rangle$ with domain $D(H^{1/2})$.

Define $\lambda(L) := \sup\{Q(f) : f \in L, \|f\| = 1\}$ for $L \subseteq D(H^{1/2})$.

<u>Thm 4.5.3:</u> Let $\mathcal{D}$ be a core for Q, i.e. $\mathcal{D} \subseteq D(H^{1/2})$ dense in the norm $|||f||| := (Q(f) + \|f\|^2)^{1/2}$,

$\lambda_n' := \inf\{\lambda(L) : L \subseteq \mathcal{D}, \dim L = n\}$

$\lambda_n'' := \inf\{\lambda(L) : L \subseteq D(H^{1/2}), \dim L = n\}$

$\Rightarrow \lambda_n = \lambda_n' = \lambda_n''$ $\forall n \geq 1$.

<u>Proof:</u> see Davies for the simple density argument. $\boxed{2}$

<u>Exercise:</u> Check that

$$\lambda_n = \sup_{\substack{L \subseteq D(H) \\ \dim L \leq n-1}} \inf_{\substack{0 \neq f \in D(H) \\ f \perp L}} \frac{\langle Hf, f\rangle}{\langle f, f\rangle}.$$

<u>Example:</u> If $\Omega \subset \mathbb{R}^n$ is contained in the cube $[0,b]^n$, Poincaré's inequality (Grubb, Thm. 4.29) says that $n\|\varphi\|_{L^2}^2 \leq \frac{b^2}{2} \sum_j \|\partial_j \varphi\|_{L^2}^2$ $\forall \varphi \in H_0^1(\Omega)$. The operator associated to the quadratic form $Q(f) = \sum_{j=1}^n \langle \partial_j f, \partial_j f\rangle$ with domain $H_0^1(\Omega)$ is the Laplace operator with Dirichlet

boundary conditions. By theorem 4.5.3, the smallest eigenvalue $\lambda_1 \geq \frac{2n}{b^2} > 0$, and the best possible constant $C(\Omega)$ in the Poincaré inequality $\|\varphi\|_{L^2}^2 \leq C(\Omega) \sum_{j=1}^{n} \|\partial_j \varphi\|_{L^2}^2 \quad \forall \varphi \in H_0^1(\Omega)$ is precisely $\frac{1}{\lambda_1}$. Note that if $\Omega \subset \tilde{\Omega}$, $H_0^1(\Omega) \subset H_0^1(\tilde{\Omega})$, so that $\lambda_j(\Omega) \geq \lambda_j(\tilde{\Omega})$ $\forall j$.

Neumann boundary conditions are associated with the form $Q(f) = \sum_{j=1}^{n} \langle \partial_j f, \partial_j f \rangle$ on $H^1(\Omega)$, and the smallest eigenvalue is $\lambda_1 = 0$ (set $f \equiv 1$). The variational characterization of the second eigenvalue gives the <u>Poincaré inequality for Neumann boundary conditions</u>,

$$\lambda_2 \|\varphi - \overline{\varphi}\|_{L^2}^2 \leq \sum_{j=1}^{n} \|\partial_j(\varphi - \overline{\varphi})\|_{L^2}^2 = \sum_{j=1}^{n} \|\partial_j \varphi\|_{L^2}^2 \quad \forall \varphi \in H^1$$

where $\overline{\varphi} = \frac{1}{|\Omega|} \int_\Omega \varphi$, the orthogonal projection of φ onto $E_{\lambda_1} = \{\text{constant functions}\}$. So the unique solvability of the Neumann problem modulo constant functions is equivalent to the above Poincaré inequality.

<u>Thm</u>: (<u>Weyl's inequality</u>)

Let Q, B, S lower bounded selfadjoint operators s.t. $Q = B + S$. Assume that $\lambda_{j-1}(B) < \inf \mathrm{EssSpec}(B)$, $\lambda_{k-1}(S) < \inf \mathrm{EssSpec}(S)$.

$\Rightarrow \lambda_{j+k-1}(Q) \geq \lambda_j(B) + \lambda_k(S)$.

<u>Proof</u>: $$\lambda_\ell(Q) = \sup_{\substack{U \subseteq \mathcal{H} \\ \dim U \leq \ell-1}} \inf_{\substack{0 \neq f \in D(\mathcal{H}) \\ f \perp U}} \frac{\langle Qf, f \rangle}{\langle f, f \rangle}.$$

Let X be the span of the eigenvectors associated to the eigenvalues $\lambda_1^{(B)}, \dots, \lambda_{j-1}^{(B)}$ of B and Y the span of the eigenvectors associated to the eigenvalues $\lambda_1^{(S)}, \dots, \lambda_{k-1}^{(S)}$ of S.

$$\text{Then} \quad \lambda_{j+k-1}(Q) \geq \inf_{\substack{0 \neq f \in D(H) \\ f \perp X+Y}} \frac{\langle Qf, f \rangle}{\langle f, f \rangle}$$

$$\overset{Q=B+S}{\geq} \inf_{\substack{0 \neq f \in D(H) \\ f \perp X+Y}} \frac{\langle Bf, f \rangle}{\langle f, f \rangle} + \inf_{\substack{0 \neq f \in D(H) \\ f \perp X+Y}} \frac{\langle Sf, f \rangle}{\langle f, f \rangle}$$

$$\overset{X+Y \supseteq X, Y}{\geq} \inf_{\substack{0 \neq f \in D(H) \\ f \perp X}} \frac{\langle Bf, f \rangle}{\langle f, f \rangle} + \inf_{\substack{0 \neq f \in D(H) \\ f \perp Y}} \frac{\langle Sf, f \rangle}{\langle f, f \rangle}$$

$$= \lambda_j(B) + \lambda_k(S). \qquad \boxtimes$$

The numerical range

Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ densely defined.

Def: $\mathrm{Num}(A) := \{ \langle Af, f \rangle : f \in D(A), \|f\| = 1 \}$ numerical range

By the spectral theorem $\overline{\mathrm{Num}(A)} = \overline{\text{convex hull}(\mathrm{Spec}(A))}$ if A is selfadjoint (or just normal) and $\overline{\mathrm{Num}(A)} \cap \{|z|=1\} = \mathrm{Spec}(A)$ if A unitary.

In general we have:

Thm: a) (Toeplitz-Hausdorff) $\mathrm{Num}(A)$ is convex.

b) If A is closed, Ω a connected component of $\mathbb{C} \setminus \overline{\mathrm{Num}(A)}$ s.t. $\lambda - A$ surjective for some $\lambda \in \Omega$ $\Rightarrow$ $\Omega \cap \mathrm{Spec}(A) = \emptyset$

and $\forall \lambda \in \Omega$: $\dfrac{1}{\mathrm{dist}(\lambda, \mathrm{Spec}(A))} \leq \|(\lambda - A)^{-1}\| \leq \dfrac{1}{\mathrm{dist}(\lambda, \mathrm{Num}(A))}$.

Corollary: a) If A is closed, $\mathbb{C} \setminus \overline{\mathrm{Num}(A)}$ connected and $\exists \lambda \in \mathbb{C} \setminus \overline{\mathrm{Num}(A)}$ s.t. $\lambda - A$ surjective $\Rightarrow$ $\mathrm{Spec}(A) \subseteq \overline{\mathrm{Num}(A)}$ and

$$\forall \lambda : \frac{1}{\mathrm{dist}(\lambda, \mathrm{Spec}(A))} \leq \|(\lambda - A)^{-1}\| \leq \frac{1}{\mathrm{dist}(\lambda, \mathrm{Num}(A))}$$

b) If A is bounded, $\mathrm{Spec}(A) \subseteq \overline{\mathrm{Num}(A)}$.

Proof: a) $\Omega = \mathbb{C} \setminus \overline{\mathrm{Num}(A)}$ in Thm, part b.

b) $\overline{\mathrm{Num}(A)} \subseteq \{ z \in \mathbb{C} : |z| \leq \|A\| \}$. But for any $|\lambda| > \|A\|$, $\|\frac{1}{\lambda} A\| < 1$, so $(\lambda - A)^{-1} = \frac{1}{\lambda}(1 - \lambda^{-1}A)$ exists (and is given by $\frac{1}{\lambda} \sum_{n=0}^{\infty} \lambda^{-n} A^n$). Also the complement of a bounded convex set is connected. Now apply a). $\square$

Proof of Thm: a) Replacing A by $\alpha A + \beta$ for suitable $\alpha, \beta \in \mathbb{C}$, the proof of convexity reduces to showing the following:

Claim: If $\|f\| = \|g\| = 1$, $\langle Af, f \rangle = 0$, $\langle Ag, g \rangle = 1$
$\Rightarrow$ $\forall \lambda \in (0,1)$ $\exists h \in \mathrm{span}\{f, g\}$ s.t. $\|h\| = 1$ and $\langle Ah, h \rangle = \lambda$.

Let $k: \mathbb{R}\times\mathbb{R} \to \operatorname{span}\{f,g\}$, $k(\Theta,s) = f + se^{i\Theta}g \neq 0 \Rightarrow$

$$\langle A k(\Theta,s), k(\Theta,s)\rangle = c_\Theta s + s^2, \quad c_\Theta = e^{i\Theta}\langle Ag,f\rangle + e^{-i\Theta}\langle Af,g\rangle$$

Because $c_\pi = -c_0$, the intermediate value thm $\Rightarrow \exists\, \alpha_0 \in [0,\pi]: c_{\alpha_0} \in \mathbb{R}$.

$F(s) := \dfrac{\langle A k(\alpha_0,s), k(\alpha_0,s)\rangle}{\| k(\alpha_0,s)\|^2}$ satisfies $F(0) = 0$, $\lim_{s\to\infty} F(s) = 1$

$\Rightarrow \exists\, s_0 \in (0,\infty): F(s_0) = \lambda$. $h := \dfrac{k(\alpha_0,s_0)}{\|k(\alpha_0,s_0)\|}$ does the job.

Before proving b) note that the left inequality in the assertion always holds:

<u>Observation:</u> Let Z be a closed operator on a Banach space and $z \notin \operatorname{spec}(Z)$

$$\Rightarrow \|(z-Z)^{-1}\| \geq \frac{1}{\operatorname{dist}(z, \operatorname{spec}(Z))}.$$

<u>Proof:</u> $w \in \mathbb{C}$, $|w-z| < \|(z-Z)^{-1}\|^{-1} \Rightarrow (w-Z) = \underbrace{(z-Z)}_{\text{invertible}}\Big(\mathbb{1} - \underbrace{(z-w)(z-Z)^{-1}}_{\|\cdot\|<1}\Big)$ invertible by Neumann series

$\Rightarrow w - Z$ invertible, i.e. $w \notin \operatorname{Spec}(Z) \Rightarrow \operatorname{dist}(z, \operatorname{Spec}(Z)) \geq \|(z-Z)^{-1}\|^{-1}$. ☒

<u>Proof of Thm b):</u> As $\operatorname{Num}(A)$ is convex, the geometric formulation of Hahn-Banach says $\overline{\operatorname{Num}(A)} = \bigcap \{H : H \supseteq \operatorname{Num}(A) \text{ closed half-plane}\}$

So replacing A by $\alpha A + \beta$, $\alpha, \beta \in \mathbb{C}$ suitable, we only have to show a result for half-planes:

<u>Claim:</u> If $\operatorname{Re}\langle Af,f\rangle \leq 0 \ \forall f \in D(A)$ and $\exists \lambda$ s.t. $\operatorname{Re}\lambda > 0$, $\lambda - A$ surjective

$\Rightarrow \operatorname{Spec}(A) \subseteq \{z : \operatorname{Re} z \leq 0\}$ and $\|(\lambda - A)^{-1}\| \leq (\operatorname{Re}\lambda)^{-1} \ \forall \operatorname{Re}\lambda > 0$.

Assume first that $\lambda > 0$, $f \in D(A) \Rightarrow$

$$(*)\quad \|(\lambda-A)f\| \geq |\langle(\lambda-A)f, \tfrac{f}{\|f\|}\rangle| = |\lambda\|f\| - \langle Af,f\rangle| \geq |\lambda\|f\| - \underbrace{\operatorname{Re}\langle Af,f\rangle}_{\leq 0}|$$

$$\geq \lambda\|f\| \Rightarrow \lambda - A \text{ injective.}$$

If $\lambda - A$ also surjective $\Rightarrow \lambda \notin \operatorname{Spec}(A)$ and $(*)$ implies $\|(\lambda-A)^{-1}\| \leq \frac{1}{\lambda}$ $(**)$

$(**)$ + Observation $\Rightarrow \operatorname{dist}(\lambda, \operatorname{Spec}(A)) \geq \|(\lambda-A)^{-1}\|^{-1} \geq \lambda$

$\Rightarrow \operatorname{spec}(A) \cap \{z \in \mathbb{C} : |z-\lambda| < \lambda\} = \emptyset$

In particular, $z - A$ surjective $\forall\, 0 < z < 2\lambda$. Induction (replacing λ by $\frac{3}{2}\lambda$) leads to $\operatorname{spec}(A) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$ and $\|(z-A)^{-1}\| \leq z^{-1} \ \forall z > 0$.

If $z = \mu + i\nu$, $\mu > 0$, then $A - i\nu$ satisfies the assumptions required for $(**)$

i.e. $\|(z-A)f\| = \|(\mu - (A - i\nu))f\| \geq \mu\|f\|$ and therefore

$\|(z-A)^{-1}\| \leq \mu^{-1} = (\operatorname{Re} z)^{-1}$. ☒

Example:

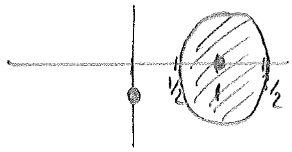
For $A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\mathrm{Num}(A) = \left\{ \begin{pmatrix}x\\ y\end{pmatrix}^t \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \overline{\begin{pmatrix}x\\ y\end{pmatrix}} \; : \; x, y \in \mathbb{C},\ |x|^2+|y|^2 = 1 \right\}$

$$= \left\{ \begin{pmatrix} e^{i\varphi}\cos(\theta) \\ e^{i\psi}\sin(\theta)\end{pmatrix}^t \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \begin{pmatrix} e^{-i\varphi}\cos(\theta) \\ e^{-i\psi}\sin(\theta)\end{pmatrix} \; : \; \theta, \varphi, \psi \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{\cos^2(\theta) + \sin^2(\theta)}_{=1} + e^{i(\varphi-\psi)} \underbrace{\cos(\theta)\sin(\theta)}_{=\frac{1}{2}\sin(2\theta)} \; : \; \theta, \varphi, \psi \in \mathbb{R} \right\}$$

$$= \left\{ 1 + \tfrac{1}{2} e^{i(\varphi-\psi)} \sin(2\theta) \; : \; \theta, \varphi, \psi \in \mathbb{R} \right\}$$

$$= \left\{ z \in \mathbb{C} \; : \; |z-1| \leq \tfrac{1}{2} \right\} =$$



$\Rightarrow$ For non selfadjoint operators, $\mathrm{Num}(A)$ can be much larger then $\mathrm{Spec}(A)$.

Schatten classes of compact operators (also known as noncommutative L^p-spaces)

(Source: Birman, Solomjak, Spectral Theory of Self-adjoint Operators in Hilbert Space)

For A compact, ≥ 0, the eigenvalues $\lambda_k(A)$ are nonnegative and accumulate (at most) at 0. It is therefore convenient to count them in decreasing order: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq 0$.

The variational characterizations are then as follows (they are easily derived from the above characterizations applied to $-A \leq 0$.):

Thm:

$$\lambda_k(A) = \min_{\substack{U \leq \mathcal{H} \\ \dim U \leq k-1}} \max_{\substack{0 \neq f \in \mathcal{H} \\ f \perp U}} \frac{\langle Af, f\rangle}{\langle f, f\rangle} = \max_{\substack{U \subset \mathcal{H} \\ \operatorname{codim} U \geq k}} \min_{\substack{0 \neq f \in \mathcal{H} \\ f \perp U}} \frac{\langle Af, f\rangle}{\langle f, f\rangle}$$

If A is compact, but not necessarily self-adjoint, define

$$s_k(A) := \lambda_k(A^*A)^{1/2} = \lambda_k(|A|).$$

$$s_k(A) = \min_{\dim U \leq k-1} \max_{f \perp U} \frac{\|Af\|}{\|f\|} = \max_{\operatorname{codim} U \geq k} \min_{f \perp U} \frac{\|Af\|}{\|f\|}$$

$$s_k(A) = \min_{\operatorname{rank} K \leq k-1} \|A-K\|_{\mathbb{B}(\mathcal{H})}$$

$\|A-K\|_{\mathbb{B}(\mathcal{H})}$ ← operator norm on $\mathcal{H}$. $=: \|A-K\|$

the k-th <u>s-number</u> or <u>singular value</u> of A.

<u>Remark</u>: Because $\|RAf\| \leq \|R\| \, \|Af\|$, we observe

$$s_k(RA) \leq \|R\| \, s_k(A), \quad s_k(AR) \leq \|R\| \, s_k(A),$$

and, in particular if U_1, U_2 unitary, $s_k(U_1 A U_2) = s_k(A)$.

Of course, also $s_k(A) \leq s_1(A) = \|A\| \ \forall k$.

Weyl's inequality: $\quad s_{m+n-1}(A_1+A_2) \leq s_m(A) + s_n(A)$ as above

Multiplicativity: $\quad s_{m+n-1}(AB) \leq s_m(A)\, s_n(B)$

Proof: Let $\operatorname{rank} K_1 \leq m-1$, $\operatorname{rank} K_2 \leq n-1$ $\Rightarrow$

$K' = K_1(AB - K_2) + AK_2$ has rank $K' \leq \operatorname{rank} K_1 + \operatorname{rank} K_2$

and $\quad AB - K = (A-K_1)(B-K_2)$.

$$\Rightarrow \quad s_{m+n-1}(AB) = \min_{\operatorname{rank} K \leq m+n-2} \|AB - K\| \leq \|AB - K'\|$$

$$\leq \|A-K_1\|\,\|B-K_2\| \qquad \forall K_1, K_2 \text{ as above}$$

Taking the minimum over K_1 and K_2, we obtain the assertion $\square$

Def: $S_p := \{ A \in \mathbb{K} : \{s_k(A)\} \in \ell^p(\mathbb{N}) \}$, $1 \leq p < \infty$, <u>p-th Schatten class</u>

$\|A\|_p := \|\{s_k(A)\}\|_{\ell^p(\mathbb{N})}$

$\Sigma_p := \{ A \in \mathbb{K} : \{s_k(A)\} \in \ell^{p,\infty}(\mathbb{N}) \}$, $1 \leq p < \infty$, <u>p-th weak Schatten class</u>

$\|A\|_{p,\infty} := \|\{s_k(A)\}\|_{\ell^{p,\infty}(\mathbb{N})}$

Here $\ell^{p,\infty}(\mathbb{N})$ are the weak-ℓ^p spaces,

$$\ell^{p,\infty}(\mathbb{N}) := \{ \{x_m\} \subseteq \mathbb{C} : \sup_{t>0} t^p \, \#\{m : |x_m| > t\} < \infty \}$$

$$= \{ \{x_m\} \subseteq \mathbb{C} : \underbrace{\sup_m |x_m|\, m^{1/p}}_{=: \|\{x_m\}\|_{\ell^{p,\infty}(\mathbb{N})}} < \infty \}.$$

Weyl's inequality implies (for $n=1$) for $C = A+B$:

- $s_m(C) \leq s_m(A) + \|B\| = s_m(A) + \|C-A\|$
- $s_m(A) = s_m(C-B) \leq s_m(C) + \|B\| = s_m(C) + \|C-A\|$

$\Rightarrow \quad |s_m(C) - s_m(A)| \leq \|C-A\|$, so the s-numbers depend continuously on A

Properties: • $\Sigma_p \subseteq S_{p+\varepsilon}$ $\quad \forall \varepsilon > 0$, $\quad S_{p_1} \subset S_{p_2}$, $\Sigma_{p_1} \subset \Sigma_{p_2}$ $\quad \forall p_1 \leq p_2$.

• $A \in \underset{\Sigma_p}{S_p} \iff A^* \in \underset{\Sigma_p}{S_p} \iff |A| := (A^*A)^{1/2} \in \underset{\Sigma_p}{S_p}$

$\iff \forall R_1, R_2 \in \mathbb{B}(\mathcal{H})$: $\quad R_1 A R_2 \in \underset{\Sigma_p}{S_p}$

• If $A, B \in \underset{\Sigma_p}{S_p} \implies s_{2n}(A+B) \leq s_{2n-1}(A+B) \leq \underset{\in\, \ell^{p(\infty)}}{s_n(A)} + \underset{\in\, \ell^{p(\infty)}}{s_n(B)}$

$\implies s_{2n}(A+B) \in \ell^{p(\infty)}$

$\implies A+B \in \underset{\Sigma_p}{S_p}$, so S_p, Σ_p vector spaces.

• S_1 = trace class operators

S_2 = Hilbert-Schmidt operators (see Ch. 1)

One often <u>defines</u> $S_\infty := \mathbb{K}$ the space of all compact operators with $\|\cdot\|_\infty$

The basic theorem is the following:

<u>Thm</u>: $\displaystyle\sum_{k \leq r} s_k(AB) \leq \sum_{k \leq r} s_k(A)\, s_k(B) \qquad \forall r \in \mathbb{N}. \qquad (*)$

We refer to Birman, Solomjak for the proof (Theorem 1 in Ch. 11.4)

<u>Corollary</u>: <u>(Hölder inequality)</u> $\quad A \in S_p$, $B \in S_q$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$

$\implies AB \in S_1$ and $\|AB\|_1 \leq \|A\|_p \|B\|_q$

<u>Proof</u>: Apply Hölder's inequality to the right hand side of $(*)$ and let $r \to \infty$. $\square$

<u>Corollary</u>: $|\mathrm{Tr}\, AB| \leq \|AB\|_1 \leq \|A\|_p \|B\|_q$ and equality is attained for some B. Therefore $\displaystyle \|A\|_p = \sup_{B \in S_q} \frac{|\mathrm{Tr}\, AB|}{\|B\|_q}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Clear from Hölder's inequality $\boxed{3}$

Cor: $p>1$, $A,B\in S_p \Rightarrow \|A+B\|_p \le \|A\|_p + \|B\|_p$

Proof: $|\mathrm{Tr}\, C(A+B)| \le |\mathrm{Tr}\, CA| + |\mathrm{Tr}\, CB|$, Take $\sup\limits_{C}$. $\boxed{4}$

Therefore $\|A\|_p = \|\{s_k(A)\}\|_{\ell^p(\mathbb{N})}$ is a norm on S_p.
Standard arguments show that S_p is complete $\Rightarrow$

Thm: S_p is a Banach space.

One can show that in fact <u>every</u> continuous linear functional on S_p, $p>1$, is given by $A \mapsto \mathrm{Tr}\, BA$ for a unique $B\in S_q$, $\frac{1}{q}+\frac{1}{p}=1$.
Therefore $S_p^* = S_q$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

As in DifFun2, $\|A\|_{p,\infty}$ is not quite a norm, satisfying only a quasi-triangle inequality $\|A+B\|_{p,\infty} \le C(\|A\|_{p,\infty} + \|B\|_{p,\infty})$ for some $C>0$. However, it can be shown that

$$\|A\|'_{p,\infty} := \sup_r r^{\frac{1}{p}-1} \sum_{k=1}^{r} s_k(A) \quad (p>1, \text{ for } p=1, \text{the prefactor is } \tfrac{1}{\log(r)})$$

defines an equivalent norm on Σ_p, turning it into a Banach space.

Example: Let A be an elliptic differential operator of order d, and assume that A is invertible. $\Rightarrow$ A^*A elliptic >0.
Ch 2: $\lambda_k(A^*A) = s_k(A)^2 \sim c\, k^{2d/\dim X}$
$\Rightarrow s_k(A^{-1}) = s_k(A)^{-1} \sim c^{-1} k^{-d/\dim X} \Rightarrow A^{-1} \in \Sigma_{\frac{\dim X}{d}}$.
We are now going to show that <u>every</u> ΨDO of order $-d$ on a compact manifold X belongs to $\Sigma_{\frac{\dim X}{d}}$.

Lemma: Let $\{A_M\}_{M\in\mathbb{N}}$, $\{A_M'\}_{M\in\mathbb{N}} \subseteq \mathbb{K}(\mathcal{H})$, $p>0$, s.t. $A_M + A_M' = A \ \forall M$.
If
- $s_k(A_M)\, k^{1/p} \xrightarrow{k\to\infty} C_M$ and $C_M \xrightarrow{M\to\infty} C_0$,
- $s_k(A_M')\, k^{1/p} \le \varepsilon_M \quad \forall k$ and $\varepsilon_M \xrightarrow{M\to\infty} 0$,

then $s_k(A)\, k^{1/p} \xrightarrow{k\to\infty} C_0$.

Proof: Write $s_j(A_M) = C_M j^{-1/p} + f_M(j) j^{-1/p}$, $f_M(j) \xrightarrow{j\to\infty} 0$.
Let $\delta\in(0,1)$ and apply Weyl's inequality with
$j = m-[\delta m]$, $k = [m\delta]+1$, $[t]$ = largest integer $\le t$:

$$\forall M \forall m: s_m(A_M + A_M')\, m^{1/p} \le s_j(A_M)\, m^{1/p} + s_k(A_M')\, m^{1/p}$$
$$\le \left(C_M + f_M(m-[\delta m])\right)\left(\frac{m}{m-[\delta m]}\right)^{1/p} + \varepsilon_M\left(\frac{m}{[\delta m]+1}\right)^{1/p}$$
$$\le \left(C_M + f_M(m-[\delta m])\right)(1-\delta)^{-1/p} + \varepsilon_M \delta^{-1/p}$$

Assuming $\varepsilon_M < 1$, let $\delta = \varepsilon_M^{p/1+p} \Longrightarrow$

$$\forall M: \limsup_{m\to\infty} s_m(A_M + A_M')\, m^{1/p} \le C_M\left(1-\varepsilon_M^{p/1+p}\right)^{-1/p} + \varepsilon_M^{p/1+p}$$

$\downarrow M\to\infty$ $\quad\swarrow M\to\infty$

$$\limsup_{m\to\infty} s_m(A)\, m^{1/p} \le C_0.$$

A similar argument using $s_m(A) \ge s_{m+[\delta m]}(A_M) - s_{[\delta m]+1}(A_M')$
shows that $\liminf_{m\to\infty} s_m(A)\, m^{1/p} \ge C_0$. $\square$

In Ch. 2 we saw $\lambda_k(A) \sim c\, k^{d/\dim(X)}$ for elliptic selfadjoint diff. operators of order $d>0$ with positive principal symbol on a compact manifold X. We conclude that $s_k(A) \sim c\, k^{d/\dim(X)}$ for elliptic DOs of order $d>0$. With a bit more work one can extend the results of ch. 2 to <u>elliptic</u> ΨDOs of order $d>0$. Recall that we even had an explicit expression for c in terms of the principal symbol of A.

The above Lemma then implies:

<u>Thm:</u> Let $A \in \Psi DO^{-d}(X)$, $d > 0$. Then $A \in \Sigma_{\frac{\dim X}{d}}$ and

$$s_k(A)\, k^{d/\dim X} \xrightarrow{k\to\infty} \frac{1}{\dim X}(2\pi)^{-\dim X} \int_X \int_{|\xi|=1} tr\left(a_d(x,\xi)^* a_d(x,\xi)\right)^{\frac{\dim X}{2d}}.$$

Note that A is <u>not</u> assumed to be elliptic.

<u>Proof:</u> $A^*A = A^*A + \frac{1}{M}\Lambda - \frac{1}{M}\Lambda$, where $\Lambda \in \Psi DO^{-2d}(X)$

$\Lambda \geq 0$ with principal symbol $[\xi]^{-2d}$. (Note that we constructed $\Lambda_s \in \Psi DO^s(X)$, with principal symbol $[\xi]^s$ in Ch. 1. One may take $\Lambda = (\Lambda_{+d}^* \Lambda_{+d} + \mathbb{1})^{-1}$. This is $\in \Psi DO^{-2d}(X)$ by the results of Ch. 1 and ≥ 0.) Then $(A^*A + \frac{1}{M}\Lambda)^{-1}$ is an elliptic selfadjoint $\Psi DO^{2d}(X)$ with positive principal symbol. Similar for $(\frac{1}{M}\Lambda)^{-1}$.

As the eigenvalues of Q^{-1} are $\lambda_k(Q)^{-1}$, the (extension of the results from Ch. 2 to ΨDO) shows that the assumptions of the Lemma are satisfied with $A_M := A^*A + \frac{1}{M}\Lambda$, $A_M' = -\frac{1}{M}\Lambda$.

$\boxed{}$