Resolvent, heat kernel and ζ-expansions

Instead of complex powers $A^{-z} = \frac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-z}(A-\lambda)^{-1}\,d\lambda$ and the associated ζ-fct $\zeta(A,z) := \operatorname{Tr} A^{-z}$ one might also be interested in traces of the resolvent, $\operatorname{tr}(A-\lambda)^{-N}$, or the heat operator, $\operatorname{tr} e^{-tA}$

We have the following relations:

Q positive, selfadjoint $\Rightarrow$ $$e^{-tQ} = \frac{i}{2\pi}\int_{\mathcal{C}'} e^{-t\lambda}(Q-\lambda)^{-1}\,d\lambda\,,\quad t>0$$
where $\mathcal{C}'$ is a curve encircling the full spectrum in the positive direction s.t. $e^{-t\lambda}$ falls off for $|\lambda|\to\infty$ out

Q positive, selfadjoint, compact resolvent $\Rightarrow$ $$Q^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-tQ}\,\Pi_0^{\perp}(Q)\,dt \qquad \operatorname{Re} z>0$$
where $\Pi_0^{\perp}(Q)$ = orthogonal projection onto eigenspaces to $\lambda \neq 0$

$$e^{-tQ}\,\Pi_0^{\perp}(Q) = \frac{1}{2\pi i}\int_{\operatorname{Re} z = c} t^{-z}Q^{-z}\,\Gamma(z)\,dz\,,\quad c>0$$

When the involved terms are trace class, one obtains relations between ζ-fcts, heat and resolvent traces. The following propositions show that knowing asymptotic expansions for one of them suffices.

(Source: G. Grubb, R.T. Seeley, Zeta and eta functions for Atiyah-Patodi-Singer operators, Journal of Geometric Analysis 6 (1996), 31–77.)

Proposition 2.9. *Suppose that f is meromorphic at 0 with Laurent expansion*

($\zeta \Leftrightarrow$ resolvent)

$$f(\lambda) = \sum_{-k}^{\infty} b_j(-\lambda)^j,\ |\lambda| \le \rho,$$

that f is holomorphic in the open sector $S_{\delta_0} = \{\lambda \in \mathbb{C} \mid |\arg\lambda - \pi| < \delta_0\}$ (for some $\delta_0 \le \pi$), and $f(\lambda) = O(|\lambda|^{-\alpha})$ for some $\alpha \in\,]0,1]$ as $\lambda \to \infty$, uniformly in each sector S_δ for $\delta < \delta_0$. Let C be a curve C_{π,r_0} as in (2.2) (a Laurent loop), with $0 < r_0 < \varrho$. Set $f_0(\lambda) = f(\lambda) - \sum_{-k}^{-1} b_j(-\lambda)^j$, and

$$\zeta(s) = \frac{i}{2\pi}\int_C \lambda^{-s} f(\lambda)\,d\lambda,\quad \operatorname{Re} s > 1-\alpha, \tag{2.27}$$

with $\lambda^{-s} = r^{-s}e^{-is\theta}$, $r>0$ and $|\theta| \le \pi$. Then

$$\zeta(s) = \frac{\sin\pi s}{\pi}\int_0^\infty r^{-s} f_0(-r)\,dr,\quad 1-\alpha < \operatorname{Re} s < 1, \tag{2.28}$$

where h_N is holomorphic for $1 - \alpha_N + \varepsilon < \operatorname{Re} s < N + 1$, and the other terms are meromorphic on $\mathbb{C}$. This gives the singularities (2.31).

To show the decay, we use the integral in (2.33) and expand on each piece of $C(\delta)$:

$$\begin{aligned}\zeta(s) &= -\frac{\mathrm{i}}{2\pi}\left(\int_0^1 + \int_1^\infty (re^{\mathrm{i}(\pi-\delta)})^{-s} f_0(re^{\mathrm{i}(\pi-\delta)}) e^{\mathrm{i}(\pi-\delta)}\, dr\right) \\ &\quad + \frac{\mathrm{i}}{2\pi}\left(\int_0^1 + \int_1^\infty (re^{\mathrm{i}(-\pi+\delta)})^{-s} f_0(re^{\mathrm{i}(-\pi+\delta)}) e^{\mathrm{i}(-\pi+\delta)}\, dr\right). \end{aligned} \tag{2.37}$$

The first integral from 0 to 1 is written as

$$\begin{aligned}\frac{-\mathrm{i}}{2\pi}\int_0^1 (re^{\mathrm{i}(\pi-\delta)})^{-s} f_0(re^{\mathrm{i}(\pi-\delta)}) e^{\mathrm{i}(\pi-\delta)}\, dr &= \frac{-\mathrm{i}}{2\pi}\int_0^1 \sum_{j=0}^{N-1} e^{\mathrm{i}(j+1-s)(\pi-\delta)} b_j r^{j-s}\, dr \\ &\quad + \int_0^1 r^{-s} e^{\mathrm{i}(\pi-\delta)(1-s)} O(r^N)\, dr \\ &= \sum_{j=0}^{N-1} \frac{-e^{\mathrm{i}(j+1-s)(\pi-\delta)} b_j}{j+1-s} \\ &\quad + e^{\mathrm{i}(\pi-\delta)(1-s)} \int_0^1 r^{-s} O(r^N)\, dr. \end{aligned}$$

Let $|\operatorname{Im} s| \geq 1$. The sum over j extends meromorphically to $\mathbb{C}$, and its terms are $O(e^{(\pi-\delta)|\operatorname{Im} s|})$ for $-\infty < C_1 \leq \operatorname{Re} s \leq C_2 < \infty$. The last term exists and is $O(e^{(\pi-\delta)|\operatorname{Im} s|})$ when $\operatorname{Re} s < N + 1$. Similar considerations hold for the other integral from 0 to 1. In the integrals from 1 to ∞ we expand as in (2.36), obtaining functions that are $O(e^{(\pi-\delta)|\operatorname{Im} s|})$ for $\operatorname{Re} s > 1 - \alpha_N + \varepsilon$. We conclude that the estimate in (2.33) extends to $1 - \alpha_N < \operatorname{Re} s < N + 1$, $|\operatorname{Im} s| \geq 1$, for arbitrarily large N. Dividing by $\sin \pi s$ we find that $\varphi(s)$ satisfies (2.32). This shows (a) $\Longrightarrow$ (b).

Conversely, assume (b). Then $f_0(-\lambda)$ is given by (2.29), and we obtain the expansion (2.30) by shifting the contour of integration past the poles of $\zeta(s)/\sin \pi s$. The remainder after all terms up to the singularity $s = 1 - \alpha_N$ is given by the integral (2.29) but with $\sigma < 1 - \alpha_N$; it is $O(|\lambda|^{-\alpha_N + \varepsilon})$ on S_δ. $\square$

Corollary 2.10. *When $f(\lambda)$ and $\zeta(s)$ are as in Proposition 2.9, then $\Gamma(s)\zeta(s)$ is meromorphic on $\mathbb{C}$ with the singularity structure*

$$\begin{aligned}\Gamma(s)\zeta(s) &\sim \sum_{j=-k}^{j=-1} \frac{-\tilde{b}_j}{s-j-1} \\ &\quad + \sum_{j=0}^{\infty}\sum_{l=0}^{m_j} \frac{\tilde{a}_{j,l}\, l!}{(s+\alpha_j-1)^{l+1}}, \qquad \tilde{b}_j = \frac{b_j}{\Gamma(-j)},\ \tilde{a}_{j,l} = \frac{a_{j,l}}{\Gamma(\alpha_j)}. \end{aligned} \tag{2.38}$$

When $\delta_0 > \frac{\pi}{2}$, *one has moreover, for any* $\delta' < \delta_0 - \frac{\pi}{2}$, *any real* C_1 *and* C_2*:*

$$|\Gamma(s)\zeta(s)| \le C'(C_1, C_2, \delta)e^{-\delta'|\operatorname{Im} s|}, \qquad \text{for } |\operatorname{Im} s| \ge 1,\ C_1 \le \operatorname{Re} s \le C_2. \tag{2.39}$$

Proof. Since $\pi(\sin \pi s)^{-1} = \Gamma(s)\Gamma(1-s)$, (2.38) results from (2.31) by multiplication by $\Gamma(1-s)^{-1}$, whose zeros cancel the poles $b_j/(s-j-1)$, $j \ge 0$. If $\delta - \pi/2 = \delta' > 0$, the estimate $|\zeta(s)| \le Ce^{(\pi-\delta)|\operatorname{Im} s|}$ shown in the proof of Proposition 2.9 (and assured by (2.32)) implies (2.39), since $\Gamma(s)$ is $O(e^{(-\frac{\pi}{2}+\varepsilon)|\operatorname{Im} s|})$ for $|\operatorname{Im} s| \ge 1$, $-\infty < C_1 \le \operatorname{Re} s \le C_2 < \infty$, any $\varepsilon > 0$ (cf., e.g., the assertion in Bourbaki [B, p. 182]):

$$|\Gamma(s)| \sim \sqrt{2\pi}\,|\operatorname{Im} s|^{\operatorname{Re} s - \frac{1}{2}} e^{-\frac{\pi}{2}|\operatorname{Im} s|} \text{ for } |\operatorname{Im} s| \to \infty, \tag{2.40}$$

valid for fixed $\operatorname{Re} s$ or $\operatorname{Re} s$ in compact intervals of $\mathbb{R}$.) □

A general result dealing with the remaining relations is the following:

Proposition 5.1. (1) *Let* $e(t)$ *be a function holomorphic in a sector* V_{θ_0} *(for some* $\theta_0 \in]0, \frac{\pi}{2}[$*),*

$$V_{\theta_0} = \{t = re^{i\theta} \mid r > 0, |\theta| < \theta_0\}, \tag{5.5}$$

such that $e(t)$ *decreases exponentially for* $|t| \to \infty$ *and is* $O(|t|^a)$ *for* $t \to 0$ *in* V_δ*, any* $\delta < \theta_0$*, for some* $a \in \mathbb{R}$*. Let* f *be the Mellin transform of* e*,*

$$f(s) = (\mathcal{M}e)(s) := \int_0^\infty t^{s-1}e(t)\,dt, \tag{5.6}$$

for $\operatorname{Re} s > -a$*. Then* $f(s)$ *is holomorphic for* $\operatorname{Re} s > -a$ *and* $f(c+i\xi)$ *is* $O(e^{-\delta|\xi|})$ *for* $|\xi| \to \infty$*, when* $c > -a$ *(uniformly for* c *in compact intervals of* $]-a, \infty[$*); and* $e(t)$ *is recovered from* $f(s)$

by the formula

$$e(t) = \frac{1}{2\pi i}\int_{\mathrm{Re}\,s=c} t^{-s} f(s)\, ds. \tag{5.7}$$

(2) *Moreover, the following properties* (a) *and* (b) *are equivalent:* (a) $e(t)$ *has an asymptotic expansion for* $t \to 0$,

$$e(t) \sim \sum_{j=0}^{\infty}\sum_{l=0}^{m_j} a_{j,l} t^{\beta_j}(\log t)^l, \quad \beta_j \nearrow +\infty,\ m_j \in \{0, 1, 2, \ldots\}, \tag{5.8}$$

uniformly for $t \in V_\delta$, *for each* $\delta < \theta_0$.

(b) $f(s)$ *is meromorphic on* $\mathbb{C}$ *with the singularity structure*

$$f(s) \sim \sum_{j=0}^{\infty}\sum_{l=0}^{m_j} \frac{(-1)^l l!\, a_{j,l}}{(s+\beta_j)^{l+1}}, \tag{5.9}$$

and for each real C_1, C_2 *and each* $\delta < \theta_0$,

$$|f(s)| \le C(C_1, C_2, \delta) e^{-\delta|\mathrm{Im}\,s|}, \ |\mathrm{Im}\,s| \ge 1,\ C_1 \le \mathrm{Re}\,s \le C_2. \tag{5.10}$$

(3) *Let* $r(\lambda)$ *take values in a Banach space, and be holomorphic in* $S_{\delta_0} = \{|\pi - \arg\lambda| < \delta_0\}$ *for some* $\delta_0 \in]\frac{\pi}{2}, \pi]$ *and meromorphic at* $\lambda = 0$ *(holomorphic for* $0 < |\lambda| < \varrho$*). Assume that as* $\lambda \to \infty$ *in* S_δ *(for* $\delta < \delta_0$*),* $\partial_\lambda^m r(\lambda)$ *is* $O(|\lambda|^{-1-\varepsilon})$ *for some* $\varepsilon > 0$ *(so that* $r(\lambda)$ *is* $O(|\lambda|^{m-1})$*). Let* θ_0 *and* θ *be such that* $]\theta - \theta_0, \theta + \theta_0[\subset]\pi - \delta_0, \frac{\pi}{2}[$, *let* $C = C_{\theta,r_0}$ *as in* (2.2) *with* $r_0 \in]0, \varrho[$, *and let*

$$e(t) = \frac{i}{2\pi}\int_C e^{-t\lambda} r(\lambda)\, d\lambda, \quad f(s) = \Gamma(s)\frac{i}{2\pi}\int_C \lambda^{-s} r(\lambda)\, d\lambda, \tag{5.11}$$

for $t \in V_{\theta_0}$ *(resp.* $\mathrm{Re}\,s > m - \varepsilon$*). Then* $e(t)$ *is exponentially decreasing for* $t \to \infty$ *in sectors* V_δ *with* $\delta < \theta_0$, *and is* $O(|t|^{-m})$ *for* $t \to 0$, *and* $f(s)$ *and* $e(t)$ *correspond to one another by* (5.6), (5.7).

Proof. (1) Note first that replacing $e(t)$ by $t^b e(t)$ replaces $f(s)$ by $f(s+b)$, so we can assume that $a > 0$ and then consider $c \ge 0$. The function $f(s)$ is holomorphic for $\mathrm{Re}\,s \ge 0$ since the integrand $t^{s-1}e(t)$ is so and has an integrable majorant there.

By a change of variables $t = e^x$, we see that $f_1(\xi) = f(i\xi)$ is the conjugate Fourier transform of $e_1(x) = e(e^x) \in L_2(\mathbb{R})$:

$$f_1(\xi) = f(i\xi) = \int_0^\infty t^{i\xi} e(t)\frac{dt}{t} = \int_{-\infty}^\infty e^{ix\xi} e(e^x)\, dx = \int_{-\infty}^\infty e^{ix\xi} e_1(x)\, dx,$$

so by Fourier's inversion formula,

$$e(t) = e_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}x\xi} f_1(\xi)\, d\xi = \frac{1}{2\pi\mathrm{i}} \int_{\operatorname{Re} s=0} t^{-s} f(s)\, ds. \tag{5.12}$$

Similarly, $f(c+\mathrm{i}\xi)$ is the conjugate Fourier transform of $e(e^x)e^{xc}$ for $c > 0$.

The hypothesis on exponential decrease of $e(t)$ in the sectors V_δ allows us to shift the path of integration in (5.6) from $t \in \mathbb{R}_+$ to $t \in e^{\mathrm{i}\delta}\mathbb{R}_+$ for $|\delta| < \theta_0$ (corresponding to a shift to $x \in \mathbb{R} + \mathrm{i}\delta$); this gives:

$$f(c+\mathrm{i}\xi) = \int_0^\infty (re^{\mathrm{i}\delta})^{c+\mathrm{i}\xi} e(re^{\mathrm{i}\delta}) \frac{dr}{r} = e^{-\delta\xi} \int_0^\infty r^{\mathrm{i}\xi} e(re^{i\delta})(re^{\mathrm{i}\delta})^c \frac{dr}{r} = e^{-\delta\xi} g(\delta, \xi, c),$$

where g is bounded as a function of $\xi \in \mathbb{R}$, locally uniformly in $c \geq 0$. Taking $\delta > 0$ for $\xi > 0$ and $\delta < 0$ for $\xi < 0$, we see that $f(c+\mathrm{i}\xi)$ decreases exponentially (like $e^{-\delta|\xi|}$) for $|\xi| \to \infty$, in any vertical strip $\{s = c + \mathrm{i}\xi \mid C_1 \leq c \leq C_2,\ \xi \in \mathbb{R}\}$ with $0 \leq C_1 \leq C_2$. Then we can also shift the integration path in (5.12) from $\operatorname{Re} s = 0$ to $\operatorname{Re} s = c$, $c \geq 0$. This shows (1).

(2) Assume now in addition (5.8). Let us first write $f(s)$ as

$$f(s) = \int_0^1 t^{s-1} e(t)\, dt + \int_1^\infty t^{s-1} e(t)\, dt.$$

The second integral defines an entire function of s. The expansion (5.8) means that

$$e(t) = \sum_{j=0}^{N-1} \sum_{l=0}^{m_j} a_{j,l} t^{\beta_j} (\log t)^l + \varrho_N(t), \qquad \varrho_N(t) = O(|t|^{\beta_N - \varepsilon}) \text{ for } t \to 0 \text{ in } V_\delta, \tag{5.13}$$

for $\varepsilon > 0$ and any positive integer N; we insert this in the first integral. Observe the formulas, valid for $\operatorname{Re} s > -\beta$,

$$\int_0^1 t^{s-1+\beta} (\log t)^l\, dt = \frac{(-1)^l l!}{(s+\beta)^l}, \qquad \int_0^\infty t^{s-1+\beta} (\log t)^l e^{-t}\, dt = \partial_s^l \Gamma(s+\beta), \tag{5.14}$$

where the cases $l > 0$ follow from the cases $l = 0$ by application of ∂_s^l. The remainder $\varrho_N(t)$ in (5.13) gives a function holomorphic for $\operatorname{Re} s > -\beta_N + \varepsilon$, and for the powers of t we use (5.14); this shows (5.9).

To show the exponential decrease of $f(s)$ on general vertical strips, one can shift the contour in (5.6) and proceed much as in the proof of Proposition 2.9. Another method, that we record here since it may be useful for further estimates, is to insert the expansion $e^t = \sum_{\nu \geq 0} \frac{1}{\nu!} t^\nu$, that gives

$$e^t t^{\beta_j} (\log t)^l = \sum_{\nu=0}^{M-1} \frac{1}{\nu!} t^{\beta_j + \nu} (\log t)^l + O(t^{\beta_j + M - \varepsilon}),$$

for any $\varepsilon > 0$ and positive integer M. Then we can write

$$e(t) = e(t)e^t e^{-t} = \left(\sum_{\beta_j+\nu<M} \sum_{l \le m_j} a_{j,l} \frac{1}{\nu!} t^{\beta_j+\nu} (\log t)^l \right) e^{-t} + \tilde{\varrho}_M(t),$$

$$\text{with } \tilde{\varrho}_M(t) = O(|t|^{M-\varepsilon}) \text{ for } t \to 0 \text{ in } V_\delta,$$

where $\tilde{\varrho}_M(t)$ is exponentially decreasing for $|t| \to \infty$ in V_δ since the other terms are so, and hence

$$f(s) = \int_0^\infty t^{s-1} \left(\sum_{\beta_j+\nu<M} \sum_{l \le m_j} a_{j,l} \frac{1}{\nu!} t^{\beta_j+\nu} (\log t)^l \right) e^{-t} dt + \int_0^\infty t^{s-1} \tilde{\varrho}_M(t)\, dt. \tag{5.15}$$

The last integral defines a function that is holomorphic for $\operatorname{Re} s > -M + \varepsilon$ and exponentially decreasing (like $e^{-\delta|\operatorname{Im} s|}$) on strips $-M + \varepsilon < C_1 \le \operatorname{Re} s \le C_2$, by (1). For the contributions from the first integral we use the second formula in (5.14) together with the fact that the gamma function $\Gamma(s)$ and its derivatives are $O(e^{(-\frac{\pi}{2}+\varepsilon')|\operatorname{Im} s|})$, any $\varepsilon' > 0$, for $|\operatorname{Im} s| \ge 1$, $-\infty < C_1 \le \operatorname{Re} s \le C_2 < \infty$, cf., e.g., [B, pp. 181–182]. This gives (5.10), completing the proof of (a) $\Longrightarrow$ (b).

Conversely, assume (b). Then $e(t)$ is given by (5.7), and we obtain the expansion (5.8) by shifting the contour of integration past the poles of $f(s)$. The remainder after all terms up to and including the singularity $s = -\beta_N$ is given by an integral like (5.7) but with $c < -\beta_N$; it is $O(|t|^{\beta_N - \varepsilon})$.

(3) That $e(t)$ defined here is exponentially decreasing for $|t| \to \infty$ in V_δ, $\delta < \theta_0$, follows since $|e^{-\lambda t}| \le e^{-\gamma|t|}$ with $\gamma > 0$ on the integration curve. The estimate for $t \to 0$ follows since

$$\int_C e^{-\lambda t} r(\lambda)\, d\lambda = (-t)^{-m} \int_C (\partial_\lambda^m e^{-\lambda t})\, r(\lambda)\, d\lambda = t^{-m} \int_C e^{-\lambda t} \partial_\lambda^m r(\lambda)\, d\lambda$$

for $t \in V_\delta$, where $e^{-\lambda t}\partial_\lambda^m r(\lambda)$ has a fixed integrable majorant for $t \to 0$. The formula (5.6) for f is shown by a complex change of variables, where we replace t by u/λ for each λ; when $\arg \lambda \in\,]0, \frac{\pi}{2}[$, the ray $\mathbb{R}_+$ is transformed to a ray Λ_λ with argument $-\arg\lambda \in\,]-\frac{\pi}{2}, 0[$, and vice versa. The integral of $u^{s-1}e^{-u}$ on such a ray is again equal to $\Gamma(s)$, as noted above. Thus (recall that $r(\lambda)$ is $O(|\lambda|^{m-1})$)

$$\begin{aligned} \int_0^\infty t^{s-1} \frac{\mathrm{i}}{2\pi} \int_C e^{-t\lambda} r(\lambda)\, d\lambda\, dt &= \frac{\mathrm{i}}{2\pi} \int_C \int_{\Lambda_\lambda} u^{s-1} \lambda^{-s} e^{-u} r(\lambda)\, du\, d\lambda \\ &= \Gamma(s) \frac{\mathrm{i}}{2\pi} \int_C \lambda^{-s} r(\lambda)\, d\lambda. \end{aligned}$$ □

<u>Cor:</u> If Q is an elliptic differential operator of order d satisfying the respective assumptions from the beginning, then

- $\operatorname{tr}(e^{-tQ}\Pi_0^+(Q)) \overset{t\to 0^+}{\sim} t^{-\frac{\dim X}{d}} \sum_{j=0}^{\infty} a_j t^{j/d}$
- If $Nd > \dim X$: $\operatorname{tr}((Q-\lambda)^{-N}) \overset{|\lambda|\to\infty}{\sim} \sum_{j=0}^{\infty} c_j (-\lambda)^{\frac{\dim X - j}{d} - N}$.