Chapter 2: Complex powers, zeta functions and spectral asymptotics

Let A be a differential operator of order $d>0$ on a compact manifold X, acting between sections of vector bundles E and F.
In local coordinates, $A = \sum_{|\alpha|\leq d} a_\alpha(x) D^\alpha$ with total symbol $a(x,\xi) = \sum_{|\alpha|\leq d} a_\alpha(x)\xi^\alpha$, where $a_\alpha(x) \in \mathrm{Hom}(E_x, F_x)$.
Let $a_j(x,\xi) = \sum_{|\alpha|=j} a_\alpha(x)\xi^\alpha$ the homogeneous term of degree j.

We denote by $\Lambda_{\varepsilon,\vartheta_0}$ the sector $\{z \in \mathbb{C} : |\arg(z)-\vartheta_0|<\varepsilon\} \cup \{0\}$.

Def: A is said to be elliptic with parameter $\lambda \in \Lambda_{\varepsilon,\vartheta_0}$ (or $\Lambda_{\varepsilon,\vartheta_0}$-elliptic)

$$\text{if } \quad \lambda \notin \sigma(a_d(x,\xi)) \quad \forall x,\xi \quad \forall \lambda \in \Lambda_{\varepsilon,\vartheta_0} \text{ s.t. } |\xi| + |\lambda|^{1/d} \neq 0.$$

Note that this is independent of the choice of local coordinates.
Let $a_d(x,\xi,\lambda) := a_d(x,\xi) - \lambda \Rightarrow a_d(x,t\xi,t^d\lambda) = t^d a(x,\xi,\lambda)$, so it is enough to check that $a_d(x,\xi,\lambda)$ is invertible for $|\xi|^{2d} + |\lambda|^2 = 1$.
We also write $a_j(x,\xi,\lambda) := a_j(x,\xi)$ $\forall j<d$ and $a_j = 0$ for $j<0$.
To understand the structure of the resolvent, we construct a parametrix to $(a(x,\xi)-\lambda)^{-1}$.

Lemma: Let A elliptic with parameter $\lambda \in \Lambda_{\varepsilon,\vartheta_0}$ for some $\varepsilon>0$, $\vartheta_0 \in [0,2\pi) \Rightarrow \forall \lambda \in \Lambda_{\varepsilon,\vartheta_0}$ there exists a classical pseudo diff. symbol $r(x,\xi,\lambda)$ of order $-d$ s.t. $r(x,\xi,\lambda) \sim \sum_{j=0}^\infty r_{-d-j}(x,\xi,\lambda)$ where

- $r_{-d}(x,\xi,\lambda) = (a_d(x,\xi)-\lambda)^{-1}$

  $$r_{-d-j}(x,\xi,\lambda) = -r_{-d}(x,\xi,\lambda) \sum_{\substack{k+\ell+|\alpha|=j \\ \ell<j}} \frac{1}{\alpha!} \partial_\xi^\alpha a_{d-k}(x,\xi) D_x^\alpha r_{-d-\ell}(x,\xi,\lambda)$$

  for $|\xi|^{2d} + |\lambda|^2 \geq 1$

- $r_{-d-j}(x,t\xi,t^d\lambda) = t^{-d-j}(x,\xi,\lambda)$ for $t \geq 1$ and $|\xi|^{2d} + |\lambda|^2 \geq 1$

- $|\partial_x^\alpha \partial_\xi^\beta r_{-d-j}(x,\xi,\lambda)| \leq C\left(1+|\xi|+|\lambda|^{1/d}\right)^{-d-j-|\beta|}$ $\forall \alpha,\beta$.

Proof: We look for r as a solution to the equation

$$1 = (a-\lambda) \# r \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \sum_{k=0}^d a_{d-k}(x,\xi,\lambda) D_x^\alpha \sum_{\ell=0}^\infty r_{-d-\ell}(x,\xi,\lambda)$$

$$\sim \sum_{j=0}^\infty \sum_{k+\ell+|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a_{d-k}(x,\xi,\lambda) D_x^\alpha r_{-d-\ell}(x,\xi,\lambda).$$

Equating terms of degree 'j' on both sides we obtain:

$j=0$: $\quad 1 = (a_d(x,\xi)-\lambda)\, r_{-d}(x,\xi,\lambda)$

$j\geq 0$: $\quad 0 = \sum_{k+l+|\alpha|=j} \frac{1}{\alpha!}\partial_\xi^\alpha a_{d-k}(x,\xi)\, D_x^\alpha r_{-d-l}(x,\xi,\lambda)$

Solving these equations for $|\xi|^{2d}+|\lambda|^2\geq 1$ we obtain the stated recursion formula for r_{-d-j}. Homogeneity for $|\xi|^{2d}+|\lambda|^2\geq 1$ and order $-d$ follow from these relations by induction over j. We now verify the estimates satisfied by r_{-d-j}.

To start with, let $\alpha=\beta=0$, $j=0$. Then

$$|r_{-d}(x,\xi,\lambda)| \leq \begin{cases} C_1, & |\xi|\leq 1,\ |\lambda|\leq 2C_3 \\ \frac{C_2}{|\lambda|-C_3}, & |\xi|\leq 1,\ |\lambda|\geq 2C_3 \\ \frac{C_4}{|\xi|+|\lambda|^{1/d}}, & |\xi|\geq 1 \end{cases} \qquad x\in X,$$

The first inequality is the continuity of r_{-d} on the compact set $|\xi|\leq 1$, $|\lambda|\leq 2C_3$, $x\in X$. The second similarly uses the boundedness of $|a_d|$ for $|\xi|\leq 1$, $x\in X$. The third inequality follows from the first by rescaling $C_1 \geq \| r_{-d}(x,t\xi,t^d\lambda)\| = t^{-d}\| r_{-d}(x,\xi,\lambda)\|$ with $t=(|\xi|^{2d}+|\lambda|^2)^{-\frac{1}{2d}}$.

The proof for general α,β and $j=0$ follows by induction using Leibniz' rule:

$$0 = \partial_{(x,\xi)}^{(\alpha,\beta)}((a_d-\lambda)\, r_{-d}) = \sum_{0<|\gamma|+|\delta|\leq|\alpha|+|\beta|} c_{\gamma\delta}\, \partial_{(x,\xi)}^{(\gamma,\delta)}(a_d-\lambda)\, \partial_{(x,\xi)}^{(\alpha-\gamma,\beta-\delta)} r_{-d} + (a_d-\lambda)\, \partial_{(x,\xi)}^{(\alpha,\beta)} r_{-d}$$

In this way, estimating $|\partial_x^\alpha \partial_\xi^\beta r_{-d}|$ is reduced to estimating strictly lower order derivatives of r_{-d} as well as derivatives of a_d. Assuming the thesis to hold for derivatives up to order $|\alpha|+|\beta|-1$, we obtain

$$|\partial_x^\alpha \partial_\xi^\beta r_{-d}| \leq C \sum_{0<|\gamma|+|\delta|\leq|\alpha|+|\beta|} c_{\gamma\delta}\, |\partial_x^\gamma \partial_\xi^\delta a_d|\, |\partial_x^{\alpha-\gamma}\partial_\xi^{\beta-\delta} r_{-d}|\, |a_d-\lambda|^{-1}$$

$$\leq \tilde{C} \sum_{\substack{|\delta|\leq d \\ 0<|\gamma|+|\delta|\leq|\alpha|+|\beta|}} c_{\gamma\delta}\, \langle\xi\rangle^{d-|\delta|}\, (1+|\xi|+|\lambda|^{1/d})^{-d-|\beta-\delta|}\, (1+|\xi|+|\lambda|^{1/d})^{-d}$$

$$
\begin{aligned}
&\overset{\varphi}{\leq} \tilde{C} \sum (1+|\xi|+|\lambda|^{1/d})^{d-|\delta|+d-|\beta-\delta|-d} \\
&\leq C_{\alpha,\beta}\,(1+|\xi|+|\lambda|^{1/d})^{-d-|\beta|}
\end{aligned}
$$

Finally, the case $j>0$ is accomplished using the same ideas combined with the recursion relation for r_{-d-j} in an induction over j. As above for each fixed j, the inequalities are proven to hold for arbitrary α, β in another induction.

As in Ch.I we can patch together these local parametrices using a partition of unity to obtain a parametrix on X. $\square$

<u>Def</u>: For $N \in \mathbb{N}$ let $R_N(\lambda) = \sum_{j=0}^{N-1} op(r_{-d-j}(\cdot,\cdot,\lambda))$.

<u>Corollary</u>: For all $s \in \mathbb{R}$ and $0 \leq t \leq d+j+|\beta|$ the operator induced by $\partial_x^\alpha \partial_\xi^\beta r_{-d-j}$ is bounded from $H^s(X,F)$ to $H^t(X,F)$ with norm $\| op(\partial_x^\alpha \partial_\xi^\beta r_{-d-j})\|_{H^s \to H^{s+t}} \leq C(1+|\lambda|^{1/d})^{-d-j-|\beta|+t}$.

In particular, we obtain $\|R_N(\lambda)\|_{H^s \to H^s} \leq C(1+|\lambda|)^{-1}$.

<u>Proof</u>: $(1+|\xi|+|\lambda|^{1/d})^{-m} \leq (1+|\xi|)^{-t}(1+|\lambda|^{1/d})^{-m+t}$ if $0 \leq t \leq m$.

The Lemma therefore says that

$$
|\partial_x^\alpha \partial_\xi^\beta r_{-d-j}(x,\xi,\lambda)(1+|\lambda|^{1/d})^{m-t}| \leq C(1+|\xi|)^{-t}, \quad m=d+j+|\beta|
$$

and the mapping properties from Ch.I imply

$$
(1+|\lambda|^{1/d})^{m-t} \| op(\partial_x^\alpha \partial_\xi^\beta r_{-d}) \|_{H^s \to H^{s+t}} \leq C, \text{ or}
$$

$$
\| op(\partial_x^\alpha \partial_\xi^\beta r_{-d}) \|_{H^s \to H^{s+t}} \leq C(1+|\lambda|)^{-d-j-|\beta|+t} \quad \square
$$

Lemma: For $N \in \mathbb{N}$; $s \in \mathbb{R}$; $-N \leq t \leq d$:

$$\| \mathbb{1} - (A-\lambda) R_N(\lambda) \|_{H^s \to H^{s+N-d+t}} \leq C \,(1+|\lambda|^{1/d})^{-d+t}.$$

Proof: From the construction of R_N we see that the symbol of $\mathbb{1} - (A-\lambda) R_N(\lambda)$ is given by a sum of terms of the form $(\partial_\xi^\alpha a_{d-k})(D_x^\alpha r_{-d-\ell})$ and λr_{-d-j}, their order being $\leq -N$. The statement now follows from the above Corollary. $\square$

Remark: The above estimate can certainly be sharpened.

"Thm 9.2:" Let A be a $\Lambda_{\varepsilon,\vartheta_0}$-elliptic differential operator, $\varepsilon_0 < \varepsilon$.

a) $A - \lambda : H^{s+d}(X) \subset H^s(X) \longrightarrow H^s(X)$ is invertible for $\lambda \in \Lambda_{\varepsilon_0,\vartheta_0}$, and $|\lambda| \geq R$.

and for any $0 \leq t \leq d$ $\quad \|(A-\lambda)^{-1}\|_{H^s \to H^{s+t}} \leq C |\lambda|^{-1+\frac{t}{d}}$.

b) $R_N(\lambda)$ converges to $(A-\lambda)^{-1}$ modulo smoothing operators as $N \to \infty$ in the sense that $\|(A-\lambda)^{-1} - R_N(\lambda)\|_{H^s \to H^{s+N-d+t}} \leq C |\lambda|^{-\frac{2d+t}{d}}$

Proof: a) By the Lemma, $\mathbb{1} - (A-\lambda) R_N(\lambda)$ converges to $\mathbb{1}$ in the operator norm on $H^s(X)$ for large λ, so $\exists\, R > 0$ $\|(A-\lambda) R_N(\lambda)\|_{H^s \to H^s} < \frac{1}{2}$ for $|\lambda| \geq R$. Then the inverse is given by $R_N(\lambda) \sum_{j=0}^{\infty} (\mathbb{1} - (A-\lambda) R_N(\lambda))^j$ and satisfies the estimate of the statement, because $R_N(\lambda)$ does so.

b) Use $(A-\lambda)^{-1} - R_N(\lambda) = (A-\lambda)^{-1} (\mathbb{1} - (A-\lambda) R_N(\lambda))$ together with the estimates of the preceding Lemma and part a) $\square$

Remark: We could have avoided discussing the N-dependence of R_N by directly choosing $R_\infty \sim \sum_{j=0}^{\infty} op(r_{-d-j})$, but this would have involved specifying the λ-dependence of the negligible $S^{-\infty}$-ambiguity contained in "$\sim$".

Def: A ray $R_\vartheta := \mathbb{R}_+ e^{i\vartheta}$ is said to be a <u>ray of minimal growth</u> for a closed operator $A : D(A) \subset X \to X$ if $\|(A-\lambda)^{-1}\|_{X\to X} \leq \frac{C}{|\lambda|}$ $\forall \lambda \in R_\vartheta$.

Cor: Assume A is an invertible $\Lambda_{\varepsilon,\vartheta_0}$-elliptic differential operator. Then A admits a ray R_ϑ of minimal growth for some $\vartheta \in (\vartheta_0-\varepsilon, \vartheta_0+\varepsilon_0)$ and $\exists\, \rho > 0 : \sigma(A) \cap B_\rho(0) = \emptyset$.

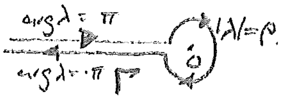
Proof: By Thm 3.2 $\|(A-\lambda)^{-1}\|_{H^s\to H^s} \leq \frac{C}{|\lambda|}$ for $\lambda \in \Lambda_{\varepsilon_0,\vartheta_0}$, $|\lambda| \geq R$.

As $d > 0$, A has compact resolvent, so there are only finitely many eigenvalues in $\{|\lambda| \leq R\}$. Choose any $\vartheta \in (\vartheta_0-\varepsilon_0, \vartheta_0+\varepsilon_0)$ s.t. $R_\vartheta \cap \sigma(A) = \emptyset$. As $0 \notin \sigma(A)$ $\|(A-\lambda)^{-1}\|_{H^s\to H^s} \leq \frac{C}{|\lambda|}$ $\forall \lambda \in R_\vartheta$. ∎

Remark: As $\sigma(A)$ is independent of $s \in \mathbb{R}$, R_ϑ can be chosen to be a ray of minimal growth for A as an operator on $H^s(X)$, independently of $s \in \mathbb{R}$.

Assumption: From now on we assume that A is invertible, $\Lambda_{\varepsilon,\pi}$-elliptic and that $(-\infty,0]$ is a ray of minimal growth for A.

Definition of A^z for $\mathrm{Re}\, z<0$:

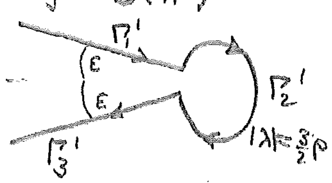
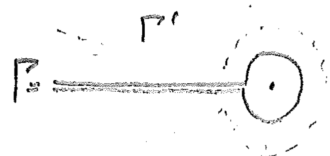
Consider $A_z := \frac{i}{2\pi} \int_\Gamma \lambda^z (A-\lambda)^{-1}\, d\lambda$, $\mathrm{Re}\, z<0$,  , $\lambda^z = |\lambda|^z e^{iz\arg\lambda}$

As $\|(A-\lambda)^{-1}\|_{L^2\to L^2} \leq \frac{C}{|\lambda|}$, the integral converges absolutely for $\mathrm{Re}\, z<0$ in the operator norm of L^2 and defines a bounded operator. Similarly on H^s $\forall s$ and on $C^\infty = \bigcap H^s$, $\mathcal{D}' = \bigcup H^s$.

Prop. 10.1: a) For $\mathrm{Re}\, z<0$, $\mathrm{Re}\, w<0$: $A_z A_w = A_{z+w}$

b) $k\in\{1,2,3,\dots\} \Rightarrow A_{-k} = (A^{-1})^k$

c) $z\mapsto A_z$ is holomorphic $\forall s\in\mathbb{R}$.
$\{\mathrm{Re}\, z<0\}\to \mathbb{B}(H^s)$

Proof: a) $\Gamma' =$  , ε small. 

$A_z = \frac{i}{2\pi}\int_{\Gamma'} \lambda^z (A-\lambda)^{-1}\, d\lambda$ (by Cauchy's integral thm.)

$$A_z A_w = \left(\frac{i}{2\pi}\right)^2 \int_{\Gamma'} d\lambda \int_\Gamma d\mu\, \underbrace{(A-\lambda)^{-1}(A-\mu)^{-1}}_{=\frac{1}{\lambda-\mu}\left((A-\lambda)^{-1}-(A-\mu)^{-1}\right)} \lambda^z \mu^w$$

"resolvent identity": Proof: multiply both sides by $(A-\lambda)(A-\mu)$.

$$= \frac{i}{2\pi}\int_{\Gamma'} d\lambda\, \lambda^z (A-\lambda)^{-1} \underbrace{\frac{i}{2\pi}\int_\Gamma \frac{\mu^w}{\lambda-\mu}\, d\mu}_{=\lambda^w \text{ (Cauchy's integral formula)}} - \left(\frac{i}{2\pi}\right)^2 \int_\Gamma d\mu (A-\mu)^{-1}\mu^w \underbrace{\int_{\Gamma'} d\lambda \frac{\lambda^z}{\lambda-\mu}}_{=0}$$

$$= A_{z+w} + 0 = A_{z+w}$$

b) $z=-1,-2,\dots \Rightarrow (re^{i\pi})^z = (re^{-i\pi})^z = r^z e^{iz\pi} \Rightarrow \int_{\arg\lambda=\pi} = -\int_{\arg\lambda=-\pi}$

$$\Rightarrow A_{-k} = \frac{i}{2\pi}\int_{|\lambda|=\rho} \lambda^{-k}(A-\lambda)^{-1}\, d\lambda = -\frac{i}{2\pi}\int_{|\mu|=\frac{1}{\rho}} \mu^k \underbrace{(A-\mu^{-1})^{-1}}_{=\mu A^{-1}(\mu-A^{-1})^{-1}} \mu^{-2}\, d\mu$$

$$= A^{-1}\underbrace{\left(-\frac{i}{2\pi}\right)\int_{|\mu|=\frac{1}{\rho}} \mu^{k-1}(\mu-A^{-1})^{-1}\, d\mu}_{=(A^{-1})^{k-1}} = (A^{-1})^k$$

since A^{-1} is bounded, $\sigma(A^{-1}) \subseteq \{|\mu|<\frac{1}{\rho}\}$ by functional calculus of bounded operators

c) Recall from DifFunc (Chapter 1):

Thm: $G \subseteq \mathbb{C}^n$ open, $f : G \times X \to \mathbb{C}$ st. a) $x \longmapsto f(z,x) \in L^1(X)$ $\forall z \in G$
(X,μ) measure space
b) $z \longmapsto f(z,x) \in \mathcal{O}(G)$ $\forall x \in X$
c) For every compact disk $K \subset G$ $\exists g_K \in L^1(X)$ $\forall z \in K$ $\forall x \in X$ $|f(z,x)| \leq g_K(x)$.

$$\longrightarrow F(z) := \int_X f(z,x)\, d\mu(x) \in \mathcal{O}(G)$$
and: $\partial_z^n F(z) = \int_X \partial_z^n f(z,x)\, d\mu(x)$ $\forall n \in \mathbb{N}_0$.

The proof is the very same for $f: G \times X \to \mathbb{B}$, $\mathbb{B}$ Banach space. To apply this, fix z_0 with $\mathrm{Re}\, z_0 < 0$, $G = B_\varepsilon(z_0)$, $\varepsilon \leq |\mathrm{Re}\, z_0|$, $X = \Gamma$ and $f(z,\lambda) = \lambda^z (A-\lambda)^{-1}$ and conclude that A_z depends holomorphically on z in $B_\varepsilon(z_0)$. ☒

Definition of A^z for all $z \in \mathbb{C}$:

Def: Let $z \in \mathbb{C}$, $k \in \mathbb{Z}$, $\mathrm{Re}\, z < k$. Set $A^z := A^k A_{z-k}$ as composition of operators on $\mathcal{D}'(M)$ (or $C^\infty(M)$).

Thm 10.1: a) A^z is well-defined (independent of k).
b) $\mathrm{Re}\, z < 0 \Rightarrow A^z = A_z$.
c) $A^z A^w = A^{z+w}$ $\forall z,w \in \mathbb{C}$.
d) $k \in \mathbb{Z} \Rightarrow A^k$ "usual" k-th powers: $A^0 = \mathrm{Id}$, $A^1 = A$, A^{-1} = inverse of A.
e) $\{\mathrm{Re}\, z < k\} \longrightarrow \mathbb{B}(H^s, H^{s-mk})$ holomorphic for all k.
$z \longmapsto A^z$

Proof: a) Let $\mathrm{Re}\, z < k < \ell$. Verify $A^k A_{z-k} = A^\ell A_{z-\ell} \Leftrightarrow A_w = A^{-p} A_{w+p}$, $p = k-\ell$, $w = z-k$.
As $\mathrm{Re}(w+p) < 0$, we can apply Prop 10.1: $A^{-p} = A_{-p}$ and $A_w = A_{-p} A_{w+p}$.
b), c), d) now follow from Prop. 10.1
e) Use $A^k : H^s(M) \to H^{s-km}(M)$ for $k \in \mathbb{Z}$, and Prop. 10.1 c). ☒

Holomorphic families of classical symbols

Def: a) Let $\mu \in \mathbb{C}$, $a \in S^{\mathrm{Re}\,\mu}_{1,0}(\Omega, \mathbb{R}^n)$. a is said to be classical of order μ

if $\quad a \sim \sum\limits_{j \geq 0} a_{\mu-j}(x,\xi)$, $a_{\mu-j}(x,t\xi) = t^{\mu-j}\, a(x,\xi) \quad \forall |\xi| \geq 1\ \forall t \geq 1$.

b) A pseudo differential operator on a manifold is classical of order $\mu \in \mathbb{C}$, if in every chart its symbol is classical of order μ.

The space of classical symbols of order μ is denoted by S^{μ}.

Def: a) Let $G \subseteq \mathbb{C}$ a domain. A family $\{a(z) : z \in G\} \subseteq S^{*}_{1,0}(\Omega, \mathbb{R}^n)$ of classical symbols is holomorphic of order $\alpha \in \mathcal{O}(G)$ if

- $a(z) \sim \sum\limits_{j \geq 0} a^{z}_{\alpha(z)-j}$ is classical of order $\alpha(z) \quad \forall z \in G$

- $$\begin{aligned} : G &\longrightarrow C^{\infty}(\Omega \times \mathbb{R}^n) \\ z &\longmapsto a^{z}_{\alpha(z)-j}(x,\xi) \end{aligned}$$ is holomorphic

- $\forall N$: $$\begin{aligned} G &\longrightarrow C^{\infty}(\Omega \times \mathbb{R}^n) \\ z &\longmapsto a(z) - \sum_{j=0}^{N-1} a^{z}_{\alpha(z)-j} =: r_N(z) \end{aligned}$$ is holomorphic and $\forall k \forall \varepsilon$: $\partial_z^k\, r_N(z) \in S^{\alpha(z)-N+\varepsilon}_{1,0}(\Omega, \mathbb{R}^n)$.

b) A family of pseudodiff. operators on a manifold is holomorphic of order $\alpha \in \mathcal{O}(G)$ if in every chart the symbol is holomorphic of order α.

A^z as a pseudodifferential operator:

Theorem: A^z is a classical pseudodifferential operator of order dz. If $\operatorname{Re} z < 0$, the classical expansion of its symbol $a^z(x,\xi)$ is given by the homogeneous terms (for $|\xi| \geq 1$)

$$a^z_{dz-j}(x,\xi) = \frac{i}{2\pi} \int_\Gamma \lambda^z \, r_{-d-j}(x,\xi,\lambda)\, d\lambda$$

where Γ is the path described above.

Proof: Due to "Thm 9.2" b) we obtain the expansion of A^z up to any finite order from the expansion of $\frac{i}{2\pi} \int_\Gamma d\lambda\, \lambda^z R_N(\lambda)$ by choosing N sufficiently large. Consequently, it suffices to note that the definition of op and Fubini's theorem imply

$$\int_\Gamma d\lambda\, \lambda^z \, op\left(r_{-d-j}(\cdot,\cdot,\lambda)\right) = op\left(\int_\Gamma d\lambda\, \lambda^z\, r_{-d-j}(\cdot,\cdot,\lambda)\right)$$

The remaining assertions follow directly from this integral representation and the homogeneity of r_{-d-j}. See Shubin, 11.2 for details. $\boxed{}$

Remark: Clearly, properties of A^z, $\operatorname{Re} z < 0$, such as holomorphy, the semigroup property and consistency with the powers of A^{-1} carry over to the symbol expansions with their composition product $\#$. Thus we may employ an analogous trick as on the operator level to extend $a^z(x,\xi)$ to a holomorphic group on $\mathbb{C}$, which is the symbols of the group of operators A^z.
Again, we refer to Shubin, Chapter 11, for details.

Conclusion: $\{A^z : z \in \mathbb{C}\}$ is a holomorphic family of order z.

The Kontsevich-Vishik trace

Recall that the trace of a pseudodifferential operator of order $< -\dim X$ could be computed as the integral of its symbol in any coordinate systems: $\mathrm{Tr}\ \mathrm{op}(a) = \int dx \int d\xi\ a(x,\xi)$.

We are going to construct an extension TR of Tr to operators of order $\in \mathbb{C} \setminus \{-\dim X + \mathbb{N}_0\}$.

Let $a(x,\xi) \in S^{\mu}$. Write $a(x,\xi) \doteq \sum_{j=0}^{N-1} a_{\mu-j}(x,\xi) + r_N(x,\xi)$, where

$a_{\mu-j}(x,t\xi) = t^{\mu-j} a_{\mu-j}(x,\xi)$, $r_N(x,\xi) \in S^{\mu-N}$, $\mathrm{Re}\,\mu - N < -\dim X$.

Then $\forall R>0$:

$$\int_{B(0,R)} a(x,\xi)\, d\xi = \sum_{j=0}^{N-1} \underbrace{\int_{B(0,R)} a_{\mu-j}(x,\xi)\, d\xi} + \underbrace{\int_{B(0,R)} \overbrace{r_N(x,\xi)}^{\in L^1(\mathbb{R}^{\dim X}_\xi)}\, d\xi}_{\xrightarrow[R\to\infty]{} \int_{\mathbb{R}^{\dim X}} r_N(x,\xi)\, d\xi}$$

$$= \underbrace{\int_{B(0,1)} a_{\mu-j}(x,\xi)\, d\xi}_{\text{independent of } R} + \underbrace{\int_{B(0,R)\setminus B(0,1)} a_{\mu-j}(x,\xi)\, d\xi}$$

$$= \int_1^R dr\, r^{\dim X - 1} \int_{|\xi|=1} a_{\mu-j}(x, r\xi)\, d\sigma(\xi)$$

$$= \underbrace{\int_1^R dr\, r^{\mu-j+\dim X-1}}_{= \begin{cases} \frac{R^{\mu-j+\dim X}-1}{\mu-j+\dim X} & \text{if } \mu-j+\dim X \neq 0 \\ \log(R) & \text{if } \mu-j+\dim X = 0 \end{cases}} \int_{|\xi|=1} a_{\mu-j}(x,\xi)\, d\sigma(\xi)$$

$$= \sum_{\substack{j=0,\\ j\neq\mu+\dim X}}^{N-1} \frac{R^{\mu-j+\dim X}}{\mu-j+\dim X} \int_{|\xi|=1} a_{\mu-j}(x,\xi) + \log(R) \int_{|\xi|=1} a_{-\dim X}(x,\xi)$$

$$- \underbrace{\int_{|\xi|\geq R} r_N}_{\xrightarrow[R\to\infty]{} 0} + \left(\sum_{j=0}^{N-1} \int_{|\xi|\leq 1} a_{\mu-j} + \int r_N - \sum_{\substack{j=0\\ j\neq\mu+\dim X}}^{N-1} \frac{1}{\mu-j+\dim X} \int_{|\xi|=1} a_{\mu-j} \right) =: -\!\!\!\!\!\!\int a(x,\xi)\, d\xi$$

R-indep. part, "renormalized integral"

Lemma: $⨍ a(x,\xi)\,d\xi$ is independent of $N > \operatorname{Re}\mu + \dim X$.

Proof: Let $N_2 > N_1 > \operatorname{Re}\mu + \dim X \Rightarrow ⨍_{N_2} a(x,\xi)\,d\xi - ⨍_{N_1} a(x,\xi)\,d\xi =$

$$= \sum_{j=N_1}^{N_2-1} \int_{|\xi|\le 1} a_{\mu-j} + \int (r_{N_2} - r_{N_1}) - \sum_{j=N_1}^{N_2-1} \frac{1}{\mu-j+\dim X} \int_{|\xi|=1} a_{\mu-j}$$

$$= \sum_{j=N_1}^{N_2-1} \int_{|\xi|\le 1} a_{\mu-j} - \int \sum_{j=N_1}^{N_2-1} a_{\mu-j} - \underbrace{\sum_{j=N_1}^{N_2-1} \frac{1}{\mu-j+\dim X} \int_{|\xi|=1} a_{\mu-j}}$$

$$\underbrace{} = -\int_1^\infty dr\; r^{\mu-j+\dim X-1} \int_{|\xi|=1} a_{\mu-j} = -\int_{|\xi|\ge 1} a_{\mu-j}$$

$$= \sum \underbrace{\left(\int_{|\xi|\le 1} - \int + \int_{|\xi|\ge 1} \right)}_{=0} = 0$$

$\square$

Lemma: $M \in GL(\dim X, \mathbb{R}) \Rightarrow ⨍ a(x, M\xi)\,d\xi = \frac{1}{|\det M|}\left(⨍ a(x,\xi)\,d\xi - \int_{|\xi|=1} a_{-\dim X}(x,\xi) \log|M^{-1}\xi|\,d\sigma(\xi) \right)$

Proof:

$$\int_{|\xi|\le R} a_{\mu-j}(x, M\xi)\,d\xi = \frac{1}{|\det M|} \int_{|M^{-1}\xi|\le R} a_{\mu-j}(x,\xi)\,d\xi$$

$$= \frac{1}{|\det M|} \Bigg(\int_{|\xi|\le 1} a_{\mu-j}(x,\xi)\,d\xi + \underbrace{\int_{\substack{|\xi|\ge 1 \\ |M^{-1}\xi|\le R}} a_{\mu-j}(x,\xi)\,d\xi}\Bigg)$$

$$\underbrace{} = \int_{|\xi|=1} a_{\mu-j}(x,\xi) \int_1^{R/|M^{-1}\xi|} r^{\mu-j+\dim X-1}\,dr\,d\sigma(\xi)$$

$$= \begin{cases} \displaystyle\int_{|\xi|=1} a_{\mu-j}(x,\xi) \frac{1}{\mu-j+\dim X}\left(\left(\frac{R}{|M^{-1}\xi|}\right)^{\mu-j+\dim X} - 1 \right) d\sigma(\xi), & \mu-j+\dim X \\ \displaystyle\int_{|\xi|=1} a_{-\dim X}(x,\xi) \log\left(R/|M^{-1}\xi| \right) d\sigma(\xi), & \mu-j = -\dim X \end{cases}$$

$\sum_j$ and taking R-indep. part $\Rightarrow$ assertion $\square$

The previous considerations trivially extend to symbols depending on (x,y,ξ).

Corollary: $\int_X dx \fint_{T_x^*X} d\xi\, \mathrm{tr}\, a(x,x,\xi)$ is independent of the choice of coordinates, if $a \in S^\mu$, $\mu \in \mathbb{C}\setminus\{-\dim X + \mathbb{N}_0\}$.

Proof: It is clear if $a \in S^{-\dim X - \varepsilon}$, (…'s) then
$$\int_X dx \fint_{T_x^*X} d\xi\, \mathrm{tr}\, a(x,x,\xi) = \int_X dx \int_{T_x^*X} d\xi\, \mathrm{tr}\, a(x,x,\xi) = \mathrm{Tr}\, op(a)$$
is independent of the choice of coordinates.

In general, write $a(x,y,\xi) = \tilde{a}(x,y,\xi) + N$, $N \in S^{-\infty}$, such that the integral kernel of $op(\tilde{a})$ is supported near the diagonal. By GG, Thm 8.1, (behavior of the symbol under changes of coordinates), the symbol of $op(\tilde{a})$ in different coordinates is
$$\tilde{a}(\kappa^{-1}(x), \kappa^{-1}(y), \phi(x,y)^{-1}\xi)\ |\det D\kappa^{-1}|\ |\det \phi^{-1}|, \quad \phi(x,x) = {}^tD\kappa^{-1}(x).$$
By the previous Lemma, for $\mu \in \mathbb{C}\setminus\{-\dim X + \mathbb{N}_0\}$
$$\int_{U_j} dx \left(\fint d\xi\ \tilde{a}(\kappa^{-1}(x), \kappa^{-1}(x), \phi(x,x)^{-1}\xi)\right) |\det D\kappa^{-1}|\ |\det \phi^{-1}|$$
$$= \int_{U_j} dx \left(\fint d\xi\ \tilde{a}(\kappa^{-1}(x), \kappa^{-1}(x), \xi)\right) |\det D\kappa^{-1}|$$
$$= \int_{\kappa^{-1}(U_j)} d\tilde{x} \fint d\xi\ \tilde{a}(\tilde{x}, \tilde{x}, \xi)$$

Summing over a covering U_j of X, we obtain the assertion. $\boxtimes$

Definition: Let $a \in S^\mu$, $\mu \in \mathbb{C}\setminus\{-\dim X + \mathbb{N}_0\}$.
$$TR\ op(a) := \int dx \fint d\xi\ \mathrm{tr}\ a(x,\xi)$$
Kontsevich-Vishik trace of $op(a)$.

Theorem: a) $TR\ op(a) = \mathrm{Tr}\ op(a)$ if $a \in S^{-\dim X - \varepsilon}$

b) $TR\ [op(a), op(b)] = 0$ if $a \in S^\mu$, $b \in S^\nu$, $\mu + \nu \notin \{-\dim X + \mathbb{N}_0\}$

c) If $A(z)$ is a holomorphic family of order α, which is not a constant $\in \{-\dim X + \mathbb{N}_0\}$, then $TR\ A(z)$ is meromorphic with poles in $z_0 \in \alpha^{-1}(\{-\dim X + \mathbb{N}_0\})$

and $\operatorname{Res}_{z=z_0} \mathrm{TR}\, A(z) = -\frac{1}{\alpha'(z_0)} \int dx \int_{|\xi|=1} d\sigma(\xi)\, \mathrm{tr}\; a^z_{-\dim X}(x,\xi)$ if $\alpha'(z_0) \neq 0$.

<u>Proof:</u> a) has been noted above

b) won't be shown here

c) follows from the explicit expression for $⨍ d\xi$. ☐

<u>Cor.:</u> (Thm 12.1, Thm 13.1)

a) The integral kernel $K_{A^z}(x,y)$ restricted to the diagonal $x=y$ extends to a meromorphic function on $\mathbb{C}$ with at most simple poles in $z_j = \frac{j-\dim X}{\operatorname{ord} A}$, $j \in \mathbb{N}_0$, and the residue in z_j is $\operatorname{Res}_{z=z_j} K_{A^z}(x,x) = -\frac{1}{\operatorname{ord} A} \int_{|\xi|=1} \mathrm{tr}\; a^z_{-\dim X}(x,\xi)\, d\sigma(\xi)$.

One can also show that this vanishes if $z_j \in \mathbb{N}_0$ and compute $K_{A^{z_j}}(x,x)$ there.

b) The function $\zeta(A,z) := \int_X K_{A^{-z}}(x,x)\, dx = \mathrm{TR}\, A^{-z}$

$$= \sum (\text{eigenvalues of } A^{-z})$$

$$= \sum (\text{eigenvalues of } A)^{-z}$$

extends meromorphically from $\operatorname{Re} z > \frac{\dim X}{\operatorname{ord} A}$ to $\mathbb{C}$ with at most simple poles at $z_j = -\frac{j-\dim X}{\operatorname{ord} A}$, $j \in \mathbb{N}_0$. Furthermore,

$$\operatorname{Res}_{z=z_j} \zeta(A,z) = \frac{1}{\operatorname{ord} A} \int_X dx \int_{|\xi|=1} \mathrm{tr}\; a^{-z}_{-\dim X}(x,\xi)\, d\sigma(\xi)$$

$$= \frac{i}{2\pi \operatorname{ord} A} \int_X dx \int_{|\xi|=1} d\sigma(\xi) \int_\Gamma \lambda^{\frac{j-\dim X}{\operatorname{ord} A}}\, r_{-\operatorname{ord} A - j}(x,\xi,\lambda)\, d\lambda$$

One can also show that this vanishes if $z_j \in -\mathbb{N}_0$ and compute $\zeta(A,z_j)$ there.

<u>Proof:</u> Follows from the above, except for the last lines. For these, we refer to Shubin. ☐

Let A be a self-adjoint, elliptic ΨDO of order d on a compact manifold X

s.t. $a_d(x,\xi) > 0 \quad \forall \xi \neq 0$.

Then A is semibounded, $A \geq -C\mathbb{1}$ on $L^2(X)$. Denote by $\lambda_1 \leq \lambda_2 \leq \dots$ its eigenvalues and by $\{\varphi_j\}$ a corresponding orthonormal basis of eigenfcts.

Wlog we are going to assume $\lambda_1 \geq 1$ (otherwise consider $A + C + 1$).

__Def:__ a)

$$E_t = \text{orthogonal projection onto } \operatorname{span}\{\varphi_j : \lambda_j \leq t\}$$

$$= \Big(u \mapsto \sum_{\lambda_j \leq t} (u,\varphi_j)_{L^2}\varphi_j\Big)$$

$$= \Big(u \mapsto \int_X e_t(x,y)\,u(y)\,dy\Big), \quad e_t(x,y) = \sum_{\lambda_j \leq t} \varphi_j(x)\overline{\varphi_j(y)} \in C^\infty(X \times X).$$

"spectral function"

b)

$$N(t) = \sum_{\lambda_j \leq t} 1 = |\{j : \lambda_j \leq t\}| = \int_X e_t(x,x)\,dx = \operatorname{Tr} E_t$$

__Note that:__ a) $e_t(x,x) = \sum_{\lambda_j \leq t} |\varphi_j(x)|^2$, $N(t)$ nondecreasing as a function of t

$e_t(x,x) = N(t) \equiv 0$ for $t < \lambda_1$

b)

$$K_{A^{-z}}(x,y) = \underbrace{\int_0^\infty t^{-z}\,d_t e_t(x,y)}_{\text{Stieltjes integral}} := \underbrace{\langle t^{-z}, \partial_t e_t(x,y)\rangle}_{\text{distributional pairing}}$$

$$\zeta(A,z) = \int_0^\infty t^{-z}\,d_t N(t) := \langle t^{-z}, \partial_t N(t)\rangle$$

__Thm 15.1 / 15.2:__ a)

$$e_t(x,x) \overset{t\to\infty}{\sim} \left[\frac{1}{\dim X}\int_{|\xi|=1} a_d^{-\frac{\dim X}{d}}(x,\xi)\,d\sigma(\xi)\right] t^{\dim X/d}$$

$$= (2\pi)^{-\frac{\dim X}{2}} \operatorname{vol}\{\xi : a_d(x,\xi) < t\} =: V_x(t)$$

b)

$$N(t) \overset{t\to\infty}{\sim} \left[\frac{1}{\dim X}\int_X\int_{|\xi|=1} a_d^{-\frac{\dim X}{d}}(x,\xi)\,d\sigma(\xi)\,dx\right] t^{\frac{\dim X}{d}}$$

$$= (2\pi)^{-\frac{\dim X}{2}} \operatorname{vol}\{(x,\xi) : a_d(x,\xi) < t\} =: V(t)$$

Corollary: $\lambda_k \overset{k\to\infty}{\sim} \left[(2\pi)^{-\frac{\dim X}{2}} \operatorname{vol}\{(x,\xi) : a_d(x,\xi) < 1\}\right]^{-\frac{d}{\dim X}} k^{d/\dim X}$

Proof of Theorem: a) $\sim$ is a direct consequence of the pole structure of $K_{A^{-z}}(x,x)$ and Ikehara's Tauberian theorem.

The equality is an elementary computation, which you can find in Shubin, Lemma 13.1.

b) $\sim$ is a direct consequence of the pole structure of $\zeta(A,z)$ and Ikehara's Tauberian theorem.

The equality is again Lemma 13.1 $\square$

The Corollary follows from:

Prop 13.1: $\alpha>0$: $N(t) \overset{t\to\infty}{\sim} c\,t^\alpha \iff \lambda_k \overset{k\to\infty}{\sim} c^{-\frac{1}{\alpha}} k^{1/\alpha}$

Proof: „$\Rightarrow$": Suppose $\forall \varepsilon>0$. $\exists t_0>0\ \forall t>t_0$: $1-\varepsilon \le N(t)\,c^{-1}t^{-\alpha} \le 1+\varepsilon$. $(*)$

Let $k_0>0$ s.t. $\lambda_{k_0} > t_0$, $\lambda_{k_0+1} > \lambda_{k_0}$.

Claim: $1-\varepsilon \le k\,c^{-1}\lambda_k^{-\alpha} \le 1+\varepsilon$

Indeed, for any $k \ge k_0$ $\exists k_1,k_2$ s.t. $k_0 \le k_1 < k \le k_2$, $\lambda_{k_1} < \lambda_{k_1+1} = \lambda_{k_2} < \lambda_{k_2+1}$

Then $N(\lambda_{k_1}) = k_1$, $N(\lambda_{k_2}) = k_2$ and $(*) \Rightarrow 1-\varepsilon \le k_{1/2}\,c^{-1}\lambda_{k_{1/2}}^{-\alpha} \le 1+\varepsilon$

Also $N(t) = k_1$ for $\lambda_{k_1} \le t < \lambda_{k_2} \Rightarrow 1-\varepsilon \le k_1 c^{-1} \underset{\lambda_{k_2}^{-\alpha}}{\underbrace{t^{-\alpha}}} \le 1+\varepsilon$

Since $k_1 < k \le k_2$, we get altogether $1-\varepsilon \le k\,c^{-1}\lambda_{k_2}^{-\alpha} \le 1+\varepsilon$.

As $\lambda_k = \lambda_{k_2}$, the claim follows.

The claim implies $(1+\varepsilon)^{-1/\alpha} c^{-1/\alpha} k^{1/\alpha} \le \lambda_k \le (1-\varepsilon)^{-1/\alpha} c^{-1/\alpha} k^{1/\alpha}$ $(**)$

„$\Leftarrow$": Choose $\varepsilon>0$, $k_0 \in \mathbb{N}$ s.t. $\forall k > k_0$ $(**)$ holds.

Let $\lambda_{k_1} \le t < \lambda_{k_2}$, k_1,k_2 as above. $(**) \Rightarrow$

$1-\varepsilon \le k_1 c^{-1}\lambda_{k_1}^{-\alpha} \le 1+\varepsilon$

$(***)$ $1-\varepsilon \le (k_1+1)\,c^{-1}\lambda_{k_2}^{-\alpha} \le 1+\varepsilon$ since $\lambda_{k_1+1} = \lambda_{k_2}$.

After possibly increasing k_0, we may assume that $c^{-1}\lambda_k^{-\alpha} \le \varepsilon$ for $k \ge k_0$. $\overset{(***)}{\Rightarrow}$ $1-2\varepsilon \le \underset{N(t)}{\underset{\|}{k_1}}\, c^{-1}\lambda_{k_2}^{-\alpha} \le 1+2\varepsilon$

so $1-2\varepsilon \le N(t)\,c^{-1}\lambda_{k_{1/2}}^{-\alpha} \le 1+2\varepsilon$ $\square$

Application to eigenfunctions:

Note that $e_{t+\varepsilon}(x,x) - e_t(x,x) = \sum_{\lambda_j = t} |\varphi_j(x)|^2$ for small $\varepsilon > 0$, so

$$|\varphi_j(x)| \leq C(x)(1+\lambda_j)^{\frac{\dim X}{2d}}.$$

If one knew an estimate $|e_t(x,x) - V_x(t)| \leq C(x) t^\alpha$, then

$$|\varphi_j(x)| \leq 2C(x)(1+\lambda_j)^\beta, \quad \beta = \tfrac{1}{2}\max\{\alpha, \tfrac{\dim X - 1}{d}\}.$$

Using refinements of Ikehara, one can get $\alpha = \frac{\dim X - 1}{d} + \varepsilon \ \forall \varepsilon > 0$.

Techniques based on Fourier integral operators (Shubin, Part III), show $\alpha = \frac{\dim X - 1}{d}$. The example of the sphere S^n (Job's talk) shows that this is optimal and cannot be improved for general manifolds. In case of the Laplacian on a manifold of constant negative curvature (Morten Risager's guest lecture), a long standing conjecture says that $\beta = \varepsilon \ \forall \varepsilon > 0$ should be true.

Actually, it is not difficult to see that $C(x)$ is bounded uniformly in x, so that one usually encounters the assertions in the form

$$\|\varphi_j\|_{L^\infty(X)} \leq C\,(1+\lambda_j)^\beta.$$

More generally one has the following sharp L^p-estimates (Sogge 1988) for the Laplacian: $\exists C \ \forall \phi$ with $\Delta\phi = \lambda\phi$:

$$\|\phi\|_{L^p(X)} \leq C\,(1+\lambda)^{\delta(p)}\,\|\phi\|_{L^2(X)},$$

where

$$\delta(p) = \begin{cases} \frac{\dim X}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{4}, & \frac{2(\dim X + 1)}{\dim X - 1} \leq p \leq \infty \\ \frac{\dim X - 1}{4}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \leq p \leq \frac{2(\dim X + 1)}{\dim X - 1}. \end{cases}$$