

Fredholm operators (mostly Thomas' talk)

Def: $T: H_1 \rightarrow H_2$ compact if $\overline{T(\mathcal{B}(0))}$ compact.

In other words, $(x_n)_{n \in \mathbb{N}}$ bounded $\Rightarrow T x_n$ has convergent subsequence.

T compact $\Rightarrow T$ is bounded.

Def: $T \in B(H_1, H_2)$ Fredholm if $\ker T$ and $\text{coker } T := H_2 / \text{im } T$ are finite dimensional.

Lemma: T Fredholm $\Rightarrow \text{im } T$ is closed.

Proof: $\tilde{T}: H_1 / \ker T \rightarrow H_2$ injective, $S: \mathbb{C}^n \cong \text{coker } T$. Then

$\tilde{T} \oplus S: H_1 / \ker T \oplus \mathbb{C}^n \rightarrow H_2$ is bijective, bounded.

$$(\tilde{T} \oplus S)(x, y) = \tilde{T}(x) + S(y)$$

$\Rightarrow \tilde{T} \oplus S$ is homeomorphism, in particular maps closed sets to closed sets.

So $\text{im } T = (\tilde{T} \oplus S)(H_1 / \ker T)$ closed. □

Sometimes Fredholm operators are defined as those operators T , for which $\ker T$ and $\ker T^*$ are finite dimensional and the image of T is closed. The latter is not automatic in this case:

If $T: H_1 \rightarrow H_2$ bounded (closed), $H_2 = \overline{\text{im } T} \oplus \ker T^*$

T Fredholm $\Rightarrow \dim \ker T = \dim \ker T^*$

So T Fredholm $\Leftrightarrow \text{im } T$ closed, $\dim \ker T < \infty$, $\dim \ker T^* < \infty$.

Def: T Fredholm. $\text{ind}(T) = \dim \ker T - \dim \ker T^*$
 $\qquad\qquad\qquad \dim \ker T = \dim \text{coker } T$.

Remark: T Fredholm $\Leftrightarrow T^*$ Fredholm

Show $\text{im } T^*$ closed $\Leftrightarrow \text{im } T$ closed (basic functional analysis),

$$\text{ind}(T^*) = \dim \ker T^* - \dim \ker T = \text{ind}(T).$$

Lemma: $T \in B^*(H_1, H_2)$, $K \in K(H_1, H_2) \Rightarrow T+K$ Fredholm.

Proof: (1) $\dim \ker(T+K) < \infty$: Show that any bounded sequence has a convergent subsequence. $x \in \ker(T+K) \Leftrightarrow Tx = -Kx$. Let $(x_n) \subset \ker(T+K)$ bounded sequence $\Rightarrow (Kx_n)$ has convergent subsequence $Kx_{n_k} = -Tx_{n_k}$

$\Rightarrow x_{n_k} \rightarrow x_0$. Thus $\ker(T+K)$ finite dim.

Same argument applies to $\ker(T^*+K^*)$.

(2) im $T+K$ is closed!: $H = \tilde{H} \oplus \ker(T+K)$. Claim: $\exists C > 0 \forall x \in \tilde{H}: \|x\| \leq C \|(T+K)x\|$

This immediately implies im $T+K$ closed.

To prove the claim, assume $\forall C > 0 \exists x \in \tilde{H}: \|x\| \geq C \|(T+K)x\|$.

Then $\exists j \nearrow \infty, \|x_j\| = 1$ s.t. $\lim_{j \rightarrow \infty} \|(T+K)x_j\| \leq \frac{1}{C} \xrightarrow{j \rightarrow \infty} 0$.

i.e. $(T+K)x_j \rightarrow 0$. But (x_n) bdd $\Rightarrow (Kx_n)$ has convergent subseq. (Kx_{n_k}) converging to 0 $\Rightarrow T x_{n_k} \xrightarrow{k \rightarrow \infty} -v \Rightarrow x_{n_k} \rightarrow -T^{-1}v = w$.

As $\|x_{n_k}\| = 1 \Rightarrow \|w\| = 1$. But $(T+K)w = \lim_{j \rightarrow \infty} (T+K)x_{j_k} = 0$

$\Rightarrow w \in \ker(T+K)$ $\not\subseteq$ contradiction to $\|w\| \leq C \|(T+K)w\|$. \square

Thm (Atkinson): $T \in \mathcal{B}(H_1, H_2)$ Fredholm $\Leftrightarrow \exists S_1, S_2 \in \mathcal{B}(H_2, H_1) \exists K_1 \in \mathcal{K}(H_1)$:
 (***) $S_1 T = I + K_1, T S_2 = I + K_2$.

Remark: $H_1 = H_2 = H$, $\mathcal{Q}(H) := \mathcal{B}(H)/\mathcal{K}(H)$ (Calkin algebra)

T is Fredholm $\Leftrightarrow [T]$ is invertible in $\mathcal{Q}(H)$.

Proof: " \Leftarrow " : It follows that the set $F(H)$ of Fredholm operators is open in $\mathcal{B}(H)$,
 \Leftrightarrow : $T: \overset{\text{H}}{H_1} \rightarrow \overset{\text{H}}{H_2}$ invertible. Define $S_2: H_2 \rightarrow H_1$ via
 $H_1 / \ker T \xrightarrow{\cong} H_2$

$S_2: H_2 \rightarrow \overset{\text{H}}{H_2} \xrightarrow{T^{-1}} \overset{\text{H}}{H_1} \hookrightarrow H_1 \Rightarrow T S_2 = \text{proj}_{H_1} = I - \text{proj}_{H_1 / \ker T}$

as $H_2 = \ker T^* \oplus \overset{\text{int}}{\underset{A_2}{\text{H}_2}}$ and $\text{proj}_{H_1 / \ker T}$ has finite rank, \Rightarrow compact.

As T^* Fredholm \Rightarrow find S_3, K_3 s.t. $T^* S_3 = I + K_3$,

$$\Rightarrow \underbrace{S_3^* T}_{S_1} = I + \underbrace{K_3^*}_{K_1}$$

" \Leftarrow " : Assume (**). $\ker T \subset \ker S_1 T = \ker I + K_1$ fin. dim.
 $\text{im } T \supset \text{im } T S_2 = \text{im } I + K_2$ fin. codim,
 $\Rightarrow T$ Fredholm. \square

- Thm: (1) $T_1 \in F(H_1, H_2), T_2 \in F(H_2, H_3) \Rightarrow T_2 T_1 \in F(H_1, H_3)$ and
 $\text{ind}(T_2 T_1) = \text{ind}(T_2) + \text{ind}(T_1)$
- (2) $T \in F(H_1, H_2), \exists C > 0 \forall S \in B(H_1, H_2), \|S\| < C \Rightarrow T+S \in F(H_1, H_2)$
 $\text{ind}(T+S) = \text{ind}(T)$,
i.e. $F(H_1, H_2)$ is open in $B(H_1, H_2)$, ind constant on connected components of $F(H_1, H_2)$.
- (3) $T \in F(H_1, H_2), K \in K(H_1, H_2) \Rightarrow T+K \in F(H_1, H_2), \text{ind}(T+K) = \text{ind}(T)$.

Def: Let V_0, \dots, V_n vector spaces, $T_j: V_j \rightarrow V_{j+1}$ linear,
Then $V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} \dots \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} V_n$ is said to be
exact if $\text{ran}(T_j) = \ker(T_{j+1})$.

Lemma: Suppose $0 \rightarrow V_1 \xrightarrow{T_1} \dots \xrightarrow{T_{n-2}} V_{n-1} \rightarrow 0$ is exact.
and $\dim V_j < \infty \forall j \Rightarrow \sum_j (-)^j \dim V_j = 0$.

Proof: Decompose $V_j = N_j \oplus Y_j$, $N_j = \ker T_j$, Y_j some complement
of N_j . Exactness $\Rightarrow T_j: Y_j \rightarrow N_{j+1}$ isomorphism
 $\Rightarrow \dim Y_j = \dim N_{j+1} \Rightarrow \dim V_j = \dim N_j + \dim Y_j$, object
 $\sum_j (-)^j \dim V_j = \sum_j (-)^j (\dim N_j + \dim N_{j+1})$

Also $\dim N_0 = 0$, $\dim V_{n-1} = \dim N_{n-1}$.

$$\Rightarrow \sum_j (-)^j \dim V_j = \sum_j (-)^j (\dim N_j + \dim N_{j+1})$$

$$= 0 + N_1 - N_1 + N_2 - N_2 + \dots + N_{n-1} - N_{n-1} = 0$$

Proof of Thm, part (1): Consider the exact sequence

$$0 \rightarrow \ker T_1 \xrightarrow{I_0} \ker T_2 T_1 \xrightarrow{T_1} \ker T_2 \xrightarrow{Q} H_2/\text{ran } T_1 \xrightarrow{T_2} H_3/\text{ran } T_1 \xrightarrow{E} H_3/\text{ran } T_2 \rightarrow 0$$

where I_0 is the inclusion of $\ker T_1$ into $\ker T_2 T_1$,

Q is the quotient map $\ker T_2 \subset H_2 \rightarrow H_2/\text{ran } T_1$,

and E maps an equivalence class mod $\text{ran } T_2 T_1$ into an equivalence class mod $\text{ran } T_2$.

Then the Lemma says

$$0 = \dim \ker T_1 - \dim \ker T_2 T_1 + \dim \ker T_2 - \dim H_2 / \text{ran } T_1 + \dim H_3 / \text{ran } T_2 T_1$$

$$= \dim H_3 / \text{ran } T_2$$

$$= \text{ind}(T_1) + \text{ind}(T_2) - \text{ind}(T_2 T_1) \Rightarrow (1)$$

(2) In case T invertible: $T+S = T(1+T^{-1}S)$. If $\|S\| < \|T^{-1}\|^{-1} := c$

$$\Rightarrow \|T^{-1}S\| < 1 \Rightarrow 1+T^{-1}S \text{ invertible with inverse}$$

$$(1+T^{-1}S)^{-1} = \sum_{k=0}^{\infty} (-T^{-1}S)^k.$$

$\Rightarrow T+S$ is invertible as product of invertible operators
and $\text{ind}(T+S) = 0 = \text{ind}(T) + \text{ind}(1+T^{-1}S)$.

The general case can be found in Geroch's book.

(3): $T \in F(H, H_2) \Rightarrow \exists S_1, S_2 : TS_1 = 1 + K_1, S_2 T = 1 + K_2 \Rightarrow (T+K)S_1 = TS_1 + KS_1 = 1 + K_1 + KS_1$,
 $S_2(T+K) = 1 + \text{compact similar}, \Rightarrow T+K \text{ Fredholm}$,
Consider $\lambda \mapsto T+\lambda K$ path of Fredholm operators

(2) $\Rightarrow \text{ind}(T+\lambda K)$ constant $\Rightarrow \text{ind}(T) = \text{ind}(T+K)$. \square

Simple proof of (2): Let R such that $RT = 1 + P_{\ker T}$ as in the proof of Atkinson's theorem $\Rightarrow R(T+S) = 1 + P_{\ker T} + RS$. For $\|S\| < \|R\|^{-1}$, $\|RS\| < 1$.
 $\Rightarrow 1+RS$ invertible $\Rightarrow (1+RS)^{-1} R(T+S) = 1 + \underbrace{(1+RS)^{-1} P_{\ker T}}_{\text{finite rank}}$. Similarly one finds a right Fredholm inverse to $T+S$. Atkinson's thm $\Rightarrow T+S$ Fredholm.

To compute $\text{ind}(T+S)$, first note that $\text{ind}(1+F) = 0$ if F is an operator of finite rank on a Hilbert space H . Indeed, let $L := \text{im } F + (\ker F)^\perp$, $\dim L < \infty$. Then $H = L \oplus L^\perp$, $(1+F)L \subset L + FL \subset L$ and $(1+F)|_{L^\perp} = 1|_{L^\perp}$, so both L and L^\perp are invariant under $1+F \Rightarrow \text{ind}(1+F) = \text{ind}(P(1+F)|_L) + \text{ind}(P(1+F)|_{L^\perp})$. As $\dim L < \infty$, linear algebra (for any matrix $\dim \ker + \dim \text{range} = 0$) says that $\text{ind}(P(1+F)|_L) = 0$. But then $\text{ind}(T) = -\text{ind}(R) + \text{ind}(1 + P_{\ker T}) = \text{ind}((1+RS)^{-1}R) + \text{ind}(1 + (1+RS)^{-1}P_{\ker T}) = \text{ind}(T+S)$. \square

Lemma (Lawson/Michelsohn III, 7.6):

$T_1, T_2 : H_1 \rightarrow H_2$ Fredholm, $\text{ind}(T_1) = \text{ind}(T_2) \Rightarrow T_1, T_2$ lie in the same connected component of Fredholm operators,

Proof: Assume $\text{ind}(T_1) = \text{ind}(T_2) \geq 0$, (otherwise consider T_1^*, T_2^* instead).

1st step: $\text{ind} T \geq 0 \Rightarrow T$ homotopic to surjective Fredholm operator.

Indeed, choose $L : \ker T \rightarrow \text{coker } T = (\text{im } T)^\perp$ surjective, linear.
Then $T + tL$, $0 \leq t \leq 1$, is a homotopy of Fredholm operators
with $T + L$ surjective.

We may thus assume T_1, T_2 surjective. Decompose $H_i = \ker T_j \oplus (\ker T_j)^\perp$,
 $j=1,2 \Rightarrow T_2^{-1} T_1 : (\ker T_1)^\perp \xrightarrow{\sim} (\ker T_2)^\perp$. Let $A : \ker T_1 \xrightarrow{\sim} \ker T_2$
any isomorphism $\Rightarrow C := A \oplus T_2^{-1} T_1 : \ker T_1 \oplus (\ker T_1)^\perp \xrightarrow{\sim} \ker T_2 \oplus (\ker T_2)^\perp$
and $T_1 = T_2 C$. We are done once we have shown:

2nd step: $B^*(H) = \{C \in B(H) \text{ invertible}\}$ is connected.

Indeed: by polar decomposition $C = U \underbrace{(C^* C)^{1/2}}_{\text{positive } \mathbb{R}}, \text{ unitary}$

$\Rightarrow C_t = U(C^* C)^{t/2}, t \in [0,1], \text{ homotopy from } C \text{ to } U$

Spectral theorem $U = \int_0^{2\pi} e^{itA} d\pi_A; U_t = \int_0^{2\pi} e^{itA} d\pi_A, 0 \leq t \leq 1,$
homotopy from U to $1 \mathbb{I}$. \square

Cor: If $H_1 = H_2$, $\text{ind} : \{\text{connected components}\}$
of Fredholm operators $\rightarrow \mathbb{Z}$ group isomorphism

Proof: $\text{ind}([T_1][T_2]) \equiv \text{ind}(T_1 T_2) = \text{ind}(T_1) + \text{ind}(T_2)$ \square

The analytic Fredholm theorem

Thm: Suppose $U \subset \mathbb{C}$ connected, open, $\{T(z)\}_{z \in U}$ family of Fredholm operators depending holomorphically on $z \in U$.

Then either

- $T(z)$ is not invertible for any $z \in U$
- or • $z \mapsto T(z)^{-1}$ is a meromorphic family of operators on H ,
the residues at the poles are finite rank operators, and $\ker T(z) \neq \{0\}$.

Proof: Let $z_1 \in U$ and Q a parametrix for $T(z_1)$, i.e.

$$Q T(z_1) = I - R \quad \text{for some } R \in K(H). \quad \text{Let } \varepsilon > 0 \text{ small.}$$

$$\Rightarrow \forall |z - z_1| < \varepsilon: \|Q(T(z) - T(z_1))\| < \frac{1}{2}$$

$$\Rightarrow \forall |z - z_1| < \varepsilon: \|I + Q(T(z) - T(z_1))\| = \|I + S(z)\| \text{ invertible.}$$

and $(I + S(z))^{-1}: B_\varepsilon(z) \rightarrow B(H)$ holomorphic.

This uses the following with $F_j = (-S)^j$:

Lemma: Let $F_j: \Omega \rightarrow B(H)$ holomorphic, $\sum_{j=0}^{\infty} \|F_j\| < \infty$
 $\Rightarrow F(z) = \sum_{j=0}^{\infty} F_j: \Omega \rightarrow B(H)$ holomorphic.

Proof: F holomorphic $\Leftrightarrow \int_{\gamma} F = 0$ \forall contractible paths γ
(Cauchy's integral theorem)

But $\int_{\gamma} F = \int_{\gamma} \sum_{j=0}^{\infty} F_j = \sum_{j=0}^{\infty} \int_{\gamma} F_j = 0 \quad \square$
dominated convergence theorem □

$$\Rightarrow (\forall) \forall |z - z_1| < \varepsilon: (I + S(z))^{-1} Q T(z) = I - \tilde{R}(z), \tilde{R}(z): B_\varepsilon(z_1) \rightarrow K(H)$$

Let F be an operator of finite rank s.t. $\|\tilde{R}_0^{-1} F\| < \frac{1}{2}$

$\forall z \in B_{\tilde{\varepsilon}}(z_1), \tilde{\varepsilon} \leq \varepsilon. \Rightarrow (I - \tilde{R}(z) + F)^{-1}$ exists $\forall z \in B_{\tilde{\varepsilon}}(z_1)$
and is analytic.

$$F \text{ finite rank} \implies \begin{aligned} &\exists \psi_1, \dots, \psi_N \in H \text{ linearly independent} \\ &\exists \phi_1, \dots, \phi_N \in H \end{aligned} : \text{S.t.}$$

$$F(\varphi) = \sum_{j=1}^N \langle \phi_j, \varphi \rangle \psi_j \quad \forall \varphi \in H.$$

$$\text{Let } \phi_n(z) := ((\mathbb{1} - \tilde{R}(z) + F)^{-1})^* \phi_n, \quad g(z) := F(\mathbb{1} - \tilde{R}(z) + F)^{-1}$$

$$\Rightarrow g(z) \varphi = \sum_{j=1}^N \langle \phi_j(z), \varphi \rangle \psi_j.$$

$$\text{Note that } \mathbb{1} - \tilde{R}(z) = \mathbb{1} + F - (\tilde{R}(z) + F) = (\mathbb{1} - F(\mathbb{1} - \tilde{R}(z) + F)).$$

$$\begin{aligned} \text{So } (\mathbb{1} - \tilde{R}(z))^{-1} \text{ exists and is holomorphic} &\iff (\mathbb{1} - F(\mathbb{1} - \tilde{R}(z) + F)^{-1})^{-1} = (\mathbb{1} - g)^{-1} \quad (\mathbb{1} - \tilde{R}(z) + F) \\ &\text{exists and is holomorphic} \\ \iff \exists 0 \neq \varphi \in H: \quad g\varphi &= \varphi \\ \iff \psi = \sum \beta_j \psi_j \neq 0 \neq \beta \in \mathbb{C}^N: \quad \sum_j \langle \phi_n(z), \psi_j \rangle \beta_j &= \beta_n \\ \iff d(z) := \det(\delta_{nj} - \langle \phi_n(z), \psi_j \rangle) &\neq 0 \end{aligned}$$

As $\phi_n(z): \mathcal{B}_E(z) \rightarrow H$ holomorphic, $d(z) \in \mathcal{O}(\mathcal{B}_E(z))$, so

either $d(z) \equiv 0$ or $Z := \{z: d(z) = 0\}$ discrete.

In the first case, (*) says that $T(z)$ is not invertible for any $z \in \mathcal{B}_E(z)$.

In the second case, $(\mathbb{1} - \tilde{R}(z))^{-1}: \mathcal{B}_E(z) \setminus Z \rightarrow \mathcal{B}(H)$ holomorphic

and the inverse can be computed by Cramer's rule (linear algebra).
involving quotients of determinants $\Rightarrow (\mathbb{1} - \tilde{R}(z))^{-1}$ meromorphic

with finite rank residues

That $\ker T(z_j) \neq \{0\}$ at a pole z_j of $T(z)^{-1}$ follows from the above
or the fact that $\text{ind } T(z_j) = 0$ if $T(z)^{-1}$ exists for some $z \in U$. \square