Chapter 1 – Pseudodifferential Operators on manifolds

WARNING! Consult e.g. Gerd's book for proofs concerning Operators on $\mathbb{R}^n$.

Recall the Fourier transform $\mathcal{F}\varphi(\xi) = \int e^{-ix\xi}\varphi(x)\,dx$, $\varphi \in S(\mathbb{R}^n)$.

The key properties $\mathcal{F}(D^\alpha\varphi) = \xi^\alpha \mathcal{F}\varphi$, $\mathcal{F}(x^\alpha\varphi) = (-D)^\alpha \mathcal{F}\varphi$

show that $\mathcal{F}: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$. It is an isomorphism with inverse

$(2\pi)^{-n}\overline{\mathcal{F}}$, $\overline{\mathcal{F}}\varphi(\xi) = \int e^{ix\xi}\varphi(x)\,dx$, and extends to $\mathcal{F}: S'(\mathbb{R}^n) \xrightarrow{\simeq} S'(\mathbb{R}^n)$

by duality: ${}_{S'}\langle \mathcal{F}u, \varphi\rangle_S := \langle u, \mathcal{F}\varphi\rangle_S \quad \forall\varphi\in S\ \forall u\in S'$.

$\mathcal{F}|_{L^2(\mathbb{R}^n)}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometric isomorphism.

Define the Sobolev spaces $H^s(\mathbb{R}^n) := \{u\in S'(\mathbb{R}^n) : \langle\xi\rangle^s \mathcal{F}u \in L^2(\mathbb{R}^n)\}$

Equivalent definitions: $s\in\mathbb{N}$:

$$H^s(\mathbb{R}^n) = \{u\in S' : D^\alpha u\in L^2\ \forall|\alpha|\le s\} = \{u\in L^2 : D^\alpha u \in L^2\ \forall|\alpha| = s\} \overset{(s\in 2\mathbb{N})}{=} \{u\in L^2 : \Delta^{s} u \in L^2\}$$

$s\in -\mathbb{N}$: $\quad H^s(\mathbb{R}^n) = H^{|s|}(\mathbb{R}^n)^*$

$s\in\mathbb{R}$: $\quad H^s(\mathbb{R}^n) = [L^2(\mathbb{R}^n), H^k(\mathbb{R}^n)]_\Theta$, $\quad k\in\mathbb{Z}$, $0\le\Theta\le1$, $s = k\Theta$.

(complex interpolation)

Fourier multipliers: $m(D)u := \mathcal{F}^{-1} m(\xi)\mathcal{F}u$

$$H^s(\mathbb{R}^n) = \{u\in S' : \langle D\rangle^s u\in L^2\}$$

We have seen in DifFun, DifFun2, that all these definitions are equivalent. Also, $H^s_{loc}(\Omega) := \{u\in D'(\Omega) : \varphi u\in H^s(\mathbb{R}^n)\ \forall\varphi\in C_c^\infty(\Omega)\}$,
$H^s_{comp}(\Omega) := \{u\in H^s(\mathbb{R}^n) : \operatorname{supp} u \Subset \Omega\}$.

Composition of Fourier multipliers is easy:

$$m(D)\tilde{m}(D) = \mathcal{F}^{-1} m\mathcal{F}\mathcal{F}^{-1}\tilde{m}\mathcal{F} = \mathcal{F}^{-1} m\tilde{m}\mathcal{F} = (m\tilde{m})(D)$$

Pseudodifferential operators allow for variable coefficients:

$$p(x,D)u(x) := \mathcal{F}^{-1}_{\xi\to x} p(x,\xi)\mathcal{F}u$$

Composition: $p(x,D)\,q(x,D) = (pq)(x,D) +$ lower order terms

Example:
$$\sum_\alpha a_\alpha D^\alpha \sum_\beta b_\beta D^\beta = \sum_{\alpha,\beta} a_\alpha b_\beta D^{\alpha+\beta} + \sum_{\alpha,\beta} \underbrace{a_\alpha[D^\alpha, b_\beta]D^\beta}_{\text{order } |\alpha|+|\beta|-1}$$

More precisely, for $\Sigma \subset \mathbb{R}^{n'}$ open, define the following symbol classes

$$S^d_{1,0}(\Sigma, \mathbb{R}^n) := \{ p \in C^\infty(\Sigma \times \mathbb{R}^n) : \forall \alpha, \beta\ \forall K \subset\subset \Sigma\ \exists C_{\alpha\beta K} : |D^\beta_x D^\alpha_\xi\, p(x,\xi)| \le C_{\alpha\beta K} \langle \xi \rangle^{d-|\alpha|} \}$$

$$= \{ p \in C^\infty(\Sigma \times \mathbb{R}^n) : \forall \alpha \beta\ \exists\ C_{\alpha\beta} \text{ continuous} : |D^\beta_x D^\alpha_\xi\, p(x,\xi)| \le C_{\alpha\beta}(x) \langle \xi \rangle^{d-|\alpha|} \}$$

$S^\infty_{1,0} := \bigcup_d S^d_{1,0}$, $S^{-\infty}_{1,0} := \bigcap_d S^d_{1,0}$.

$S^d(\Sigma, \mathbb{R}^n) := \{ p \in S^d_{1,0}(\Sigma, \mathbb{R}^n) : \exists \{p_{d-\ell}\}_{\ell=0}^\infty \subseteq C^\infty :$

① $p_{d-\ell}(x, t\xi) = t^{d-\ell} p_{d-\ell}(x,\xi) \quad \forall t \ge 1\ \forall |\xi| \ge 1$
(homogeneity for $|\xi| \ge 1$)

② $\forall N : p(x,\xi) - \sum_{\ell=0}^{N-1} p_{d-\ell}(x,\xi) \in S^{d-N}_{1,0}$
(classical expansion) $\}$

$S^\infty = \bigcup_d S^d$, $S^{-\infty} = \bigcap_d S^d$.

Condition ② is symbolically written as $p \sim \sum_\ell p_{d-\ell}$.

<u>Ex</u>: $\langle \xi \rangle = (1+|\xi|^2)^{1/2} = |\xi| \cdot (1+|\xi|^{-2})^{1/2}$

$$= |\xi| \sum_{n=0}^\infty \binom{1/2}{n} |\xi|^{-2n} = [\xi] \sum_{n=0}^\infty \binom{1/2}{n} [\xi]^{-2n} + \psi(\xi)$$

where $[\xi] \in C^\infty$ w/ $[\xi] = |\xi|$ for $|\xi| \ge 1$
and $\psi \in C^\infty_c \subseteq S^{-\infty}$. $\Longrightarrow \langle \xi \rangle \in S^d$

p_d is called the <u>principal symbol</u> of p.

Often $S^d_{1,0}$ is denoted by S^d and S^d by S^d_{cl}.

One can also define matrix-valued symbols = matrices whose entries belong to the above symbol classes.

In DifFun2 we considered a more restricted class of symbols, where for which $C_{\alpha\beta}(x)$ was bounded. For those classes, we could show stronger results, which imply most of the results we state for $S^d_{1,0}$ by multiplying $p \in S^d_{1,0}$ by cut-off fcts.

More generally one defines $S^d_{\rho,\delta}(\Sigma, \mathbb{R}^k) := \{ p : |D_x^\beta D_\xi^\alpha p| \le C_{\alpha\beta K} \langle \xi \rangle^{d-\rho|\alpha|+\delta|\beta|} \}$

The formal sum $\sum_{\ell=0}^{\infty} p_{d-\ell}$ does not converge in general.
However, for any sequence $p_{m_j} \in S^{m_j}_{1,0}$, $m_j \downarrow -\infty$, there exists $p \in S^{m_0}_{1,0}$ s.t. $p \sim \sum p_{m_j}$: <u>Proof</u>: Set $p(x,\xi) = \sum p_{m_j}(x,\xi)\,(1-\chi(\varepsilon_j \xi))$ where $\varepsilon_j \downarrow 0$ rapidly. This was an exercise in DifFun 2.

<u>Key</u>: $p \sim 0 \iff p \in S^{-\infty}$, so p is determined from the asymptotic expansion up to an element $\in S^{-\infty}$.

<u>Properties</u>:

- $S^d_{1,0}$, S^d are Fréchet spaces

  (seminorms: best constants $C_{\alpha\beta K}$ in the definition of $S^d_{1,0}$ $+ \forall N$ best constants in (2))

- $p \in S^d_{1,0}$, $q \in S^{d'}_{1,0}$ $\Rightarrow$ $p+q \in S^{\max\{d,d'\}}_{1,0}$

  $pq \in S^{d+d'}_{1,0}$

  $|p| \ge C\langle\xi\rangle^d \Rightarrow p^{-1} \in S^{-d}_{1,0}$

  Similar for S^d, as long as $p+q$, pq are classical.

- For $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, $p \in S^d_{1,0}(\Omega_1 \times \Omega_2, \mathbb{R}^n)$ induces a continuous operator $op(p) : H^{s+d}_{comp}(\Omega_2) \to H^s_{loc}(\Omega_1)$ via

  $$op(p)\,u(x) := \lim_{\varepsilon \to 0} \int e^{i(x-y)\xi} \chi(\varepsilon\xi)\, p(x,y,\xi)\, u(y)\, dy\, d\xi$$

  By Thm 6.18 $\mathcal{E}'(\Omega_2) \subseteq \bigcup_t H^t_{comp}(\Omega_2)$, so that

  $op(p) : \mathcal{E}'(\Omega_2) \longrightarrow \mathcal{D}'(\Omega_1)$ continuously.

  Its adjoint is $op(p(x,y,\xi))^* = op(p(y,x,\xi)^*)$, because the kernel of the adjoint is obtained by switching the roles of x and y and taking complex conjugates.

From now on we study pseudodifferential operators on a fixed open $\Omega \subset \mathbb{R}^n$, i.e. $op\left(S^{\infty}_{1,0}(\Omega, \mathbb{R}^n)\right)$.

- kernel estimates: The results of DifFun2 show that these operators have a distributional kernel $K \in \mathcal{D}'(\Omega \times \Omega)$, i.e. $\forall u, v \in C^\infty_c(\Omega)$:

$$\langle op(p) u, v \rangle = \langle K, u \otimes v \rangle.$$

For $op(p) = \sum a_\alpha(x) \partial^\alpha$, $K = \sum \pm a_\alpha(x) \partial^\alpha \delta_{x-y}$ is supported on the diagonal $\operatorname{diag}(\Omega) := \{(x,x) \in \Omega \times \Omega : x \in \Omega\}$. For any $p \in S^d(\Omega, \mathbb{R}^n)$,

$$K(x,y) = k(x, x-y) = \mathcal{F}^{-1}_{\xi \to x-y}\, p(x,\xi) \qquad (x \neq y)$$

is smooth as a function on $\Omega \times \Omega \setminus \operatorname{diag}(\Omega)$ and satisfies $\forall\, n+m+|\alpha|+N>0\ \forall z \neq 0$: $\left|\partial_x^\beta \partial_z^\alpha k(x,z)\right| \leq C_{\alpha\beta N}(x)\, |z|^{-n-m-|\alpha|-N}$

Calderon-Zygmund theory (DifFun2) implies that $op(p): W^{s+d,p}_{comp} \to W^{s,p}_{loc}$ acts boundedly between L^p-Sobolev spaces for $p \in (1,\infty)$.

- Composition: Unlike for the globally estimated symbols in DifFun2, the composition of operators in $op(S^\infty_{1,0}(\Omega, \mathbb{R}^n))$ need not be defined due to the possible growth in x. Even if it is, the result need not be $\in op(S^\infty(\Omega, \mathbb{R}^n))$.

Def.: A negligible (smoothing, regularizing) operator is an integral operator with kernel $\in C^\infty$.

By definition, R is negligible $\iff$ R^* is negligible.
The negligible operators are precisely the operators in $op(S^{-\infty}(\Omega \times \Omega, \mathbb{R}^n))$.
Given a smooth kernel K, a corresponding symbol is given by

$$r(x,y,\xi) = e^{i(y-x)\xi} K(x,y)\, \chi(\xi), \quad \chi \in C^\infty_c(\mathbb{R}^n), \ \int \chi = 1.$$

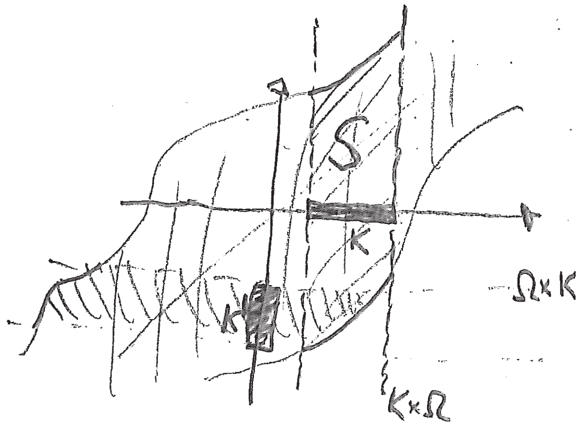
By the above mapping properties, such operators map $\mathcal{E}'(\Omega) \subseteq \bigcup_t H^t_{comp}(\Omega) \longrightarrow \bigcap_s H^s_{loc}(\Omega) \subseteq C^\infty(\Omega)$.

Def: A <u>properly supported operator</u> is an (integral) operator P s.t.

$$\forall K \subset\subset \Omega \ \exists K' \subset\subset \Omega : \quad P\, \mathcal{D}'_K(\Omega) \subseteq \mathcal{D}'_{K'}(\Omega).$$
$$P^* \mathcal{D}'_K(\Omega) \subseteq \mathcal{D}'_{K'}(\Omega)$$

Equivalently, if $S \subset \Omega \times \Omega$ is the support of the distributional kernel of P,

$S \cap (\Omega \times K)$ and $S \cap (K \times \Omega)$ are compact for all $K \subset\subset \Omega$.



"properly supported = finite width of support"

e.g. differential operators, for which $S \subseteq \mathrm{diag}(\Omega)$.

By definition, P is properly supported $\Leftrightarrow$ P^* is properly supported.
Let $P \in \mathrm{op}(S^{\infty}(\Omega, \mathbb{R}^n)) + \mathrm{op}(S^{-\infty}(\Omega \times \Omega, \mathbb{R}^n))$.
As $P, P^* : C_c^\infty \to C_c^\infty$, we obtain $P = P^{**} : \mathcal{D}' \to \mathcal{D}'$ by duality.

$$\mathcal{E}' \to \mathcal{E}' \quad \rightsquigarrow \quad P : C^\infty \to C^\infty$$

$$\forall p \in (1,\infty): \quad W^{s+d,p}_{comp} \to W^{s,p}_{comp} \quad \rightsquigarrow \quad P : W^{s+d,p}_{loc} \to W^{s+d,p}_{loc}$$

Similarly, if P is properly supported and R negligible, then $P \circ R$ and $R \circ P$ are negligible.

<u>**Thm 7.10:**</u> $p \in S^{\infty}(\Omega, \mathbb{R}^n) \Rightarrow \exists q \in S^{\infty}(\Omega, \mathbb{R}^n)\ \exists r \in S^{-\infty}(\Omega \times \Omega, \mathbb{R}^n)$:

- $\mathrm{op}(q)$ is properly supported
- $\mathrm{op}(p) = \mathrm{op}(q) + \mathrm{op}(r)$

<u>Proof:</u> Cut-off the kernel to a strip of finite width around $\mathrm{diag}(\Omega)$.
$\Rightarrow$ properly supported part $\mathrm{op}(q)$
By the kernel estimates, the remainder $\in C^\infty$ $\Rightarrow$ negligible part $\mathrm{op}(r)$ $\boxtimes$

Thm 7.13: (I) Let $S := \left(S^{\infty}_{1,0}(\Omega, \mathbb{R}^n) + S^{-\infty}(\Omega\times\Omega, \mathbb{R}^n)\right) \Big/ S^{-\infty}(\Omega\times\Omega,\mathbb{R}^n)$.

$S^{\mathbb{Z}}_{cl} := \left(\bigcup_{d\in\mathbb{Z}} S^{d}(\Omega, \mathbb{R}^n) + S^{-\infty}(\Omega\times\Omega, \mathbb{R}^n)\right) \Big/ S^{-\infty}(\Omega\times\Omega,\mathbb{R}^n)$

$\Longrightarrow$ $op(S)$ and $op(S^{\mathbb{Z}}_{cl})$ are $*$-algebras:

Given $op(p) = op(q) + op(r)$, $op(\tilde p) = \underbrace{op(\tilde q)}_{\text{properly supported}} + \underbrace{op(\tilde r)}_{\text{negligible}}$, $[op(p)]\circ[op(\tilde p)] := [op(q)\circ op(\tilde q)]$.

In particular, the composition $op(p)\circ op(\tilde p)$ of two pseudo-diff. operators (one of them properly supported) is again a pseudo differential operator $op(c)$, $c \in S^{\infty}_{1,0}(\Omega,\mathbb{R}^n) + S^{-\infty}(\Omega\times\Omega,\mathbb{R}^n)$, and
$$c(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} D^{\alpha}_{\xi} p(x,\xi)\, \partial^{\alpha}_{x} \tilde p(x,\xi) =: p \# \tilde p$$
The composition of classical operators is again classical.

<u>About the proof</u>: ① As (negligible) $\circ$ (properly supported) and (properly supported) $\circ$ (negligible) are both negligible, it is enough to show that the composition of properly supported operators has the described properties.

② Gerd Grubb does this with a trick:

– If $p \in S^{d}_{1,0}(\Omega\times\Omega, \mathbb{R}^n) \Longrightarrow \exists\, p_1, p_2 \in S^{d}_{1,0}(\Omega, \mathbb{R}^n)$ s.t. $op(p(x,y,\xi)) \sim op(p_1(x,\xi)) \sim op(p_2(y,\xi))$

(This also establishes: <u>Theorem 7.13 (II)</u>: $\exists\, p_3 \in S^{d}_{1,0}(\Omega,\mathbb{R}^n)$: $op(p)^* \sim op(p_3)$, $p_3(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial^{\alpha}_{x} D^{\alpha}_{\xi} p(x,\xi)^*$.)

Error in $op(\tilde p_2)$ in class

Then $op(p)\circ op(\tilde p) \sim op(p(x,\xi)) \circ op(\tilde p_2(y,\xi))$

$= \left(\mathcal{F}^{-1}_{\xi\to x}\, p(x,\xi)\, \underbrace{\mathcal{F}_{x\to\xi}\right)\left(\mathcal{F}^{-1}_{\xi\to x}}_{=\mathbb{1}}\, \mathcal{F}_{y\to\xi} \tilde p_2(y,\xi)\right)$

$= \mathcal{F}^{-1}_{\xi\to x}\, p(x,\xi)\, \mathcal{F}_{y\to\xi} \tilde p_2(y,\xi) = \mathcal{F}^{-1}_{\xi\to x} \mathcal{F}_{y\to\xi}\, p(x,\xi)\tilde p_2(y,\xi)$

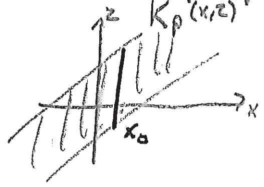
$\sim op\left((p(x,\xi)\tilde p_2(y,\xi))_1 \right)$ $\qquad = op(p(x,\xi), \tilde p_2(y,\xi))$

This works fine for our "simple" classes of symbols.

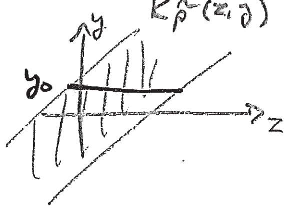
(3) Alternatively, one can use the (slightly more robust) result from DifFun2 for globally estimated symbols:

Let $op(p)$, $op(\tilde{p})$ properly supported.

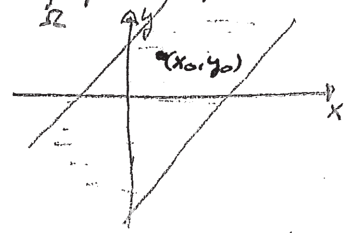
kernel of $op(p)$: $K_p(x,z)$



kernel of $op(\tilde{p})$: $K_{\tilde{p}}(z,y)$



composition: $\int_\Omega K_p(x,z)\, K_{\tilde{p}}(z,y)\, dz$



So to compute the kernel of the composition in a point or a bounded subset of $\Omega \times \Omega$, we only need to know $op(p)$, $op(\tilde{p})$ on a compact subset $\tilde{\Omega} \subset \Omega$, i.e. $\chi_{\tilde{\Omega}}\, op(p)\, \chi_{\tilde{\Omega}}$ and $\chi_{\tilde{\Omega}}\, op(\tilde{p})\, \chi_{\tilde{\Omega}}$ for some $\chi_{\tilde{\Omega}} \in C_c^\infty(\Omega)$, $\chi_{\tilde{\Omega}}|_{\tilde{\Omega}} = 1$. But these truncated operators belong to the class considered in DifFun2, so their composition is a pseudodifferential operator with the desired symbol.

You can try to work out the details yourself (see DifFun2-notes!)

(4) Both in Gerd's and the DifFun2-approach, the strategy is to write down the integral kernel (of $op(p(x,y,\xi))$ resp. $op(p) \circ op(\tilde{p})$) explicitly and to analyze it using Taylor expansions. The Taylor expansion gives the asymptotic expansion of the symbol, and the main work is to show that the error from the Taylor expansion is of order $-\infty$.

More specifically: The kernel of $op(p) \circ op(\tilde{p})$ is (if $p, \tilde{p}$ compactly supported)

$$\iiint p(x,\eta)\, \tilde{p}(z,\xi)\, e^{i(x-z)(\eta-\xi)}\, dz\, d\eta\; e^{i(x-y)\xi}\, d\xi$$

symbol is

$$\begin{aligned} c(y,\xi) &= \iint p(x,\eta)\, \tilde{p}(z,\xi)\, e^{i(x-z)(\eta-\xi)}\, dz\, d\eta \\ &= \int \underbrace{p(x,\xi+\eta)}_{= \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi)\, \eta^\alpha + R} \mathcal{F}_{z\to\eta} b(z,\xi)\, e^{ix\eta}\, d\eta \\ &= \sum_{|\alpha|\le N} \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) \underbrace{\int \eta^\alpha\, \mathcal{F}_{z\to\eta} b(z,\xi)\, e^{ix\eta}\, d\eta}_{= i^{-|\alpha|} \partial_x^\alpha b(x,\xi)} + \tilde{R} \end{aligned}$$

In Gerd's approach, $op(p(x,y,\xi))$ has the kernel

$$\int e^{i(x-y)\xi}\, p(x,y,\xi)\, d\xi = \int e^{i(x-y)\xi} \sum_{|\alpha|<N} \frac{1}{\alpha!} (y-x)^\alpha \partial_y^\alpha p(x,x,\xi)\, d\xi + \tilde{R}$$

$$= \sum_{|\alpha|<N} \frac{1}{\alpha!} \int e^{i(x-y)\xi} (-i\partial_\xi)^\alpha \partial_y^\alpha p(x,x,\xi)\, d\xi + \tilde{R}$$

$\boxed{\ }$

Theorem 7.13 allows us to prove the boundedness results on Sobolev spaces which we mentioned before. They are a consequence of:

<u>Theorem</u>: Let $p \in S^0_{1,0}(\Omega, \mathbb{R}^n)$ s.t. the kernel of $op(p)$ is compactly supported. Then $op(p)$ is bounded on $L^2(\mathbb{R}^n)$.

We show:

<u>Theorem</u>: $P := op(p) \in op(S^0_{1,0}(\Omega, \mathbb{R}^n))$ properly supported s.t. $\forall K \subset\subset \Omega$

$$\overline{\lim_{\substack{|\xi|\to\infty \\ x\in K}}} |p(x,\xi)| < M \implies \exists R \in S^{-\infty} \text{ properly supported}$$

s.t. $\|op(p)u\|^2_{L^2} \leq M^2 \|u\|^2_{L^2} + (Ru, u)$.

If the kernel of $op(p)$ is comp. supp., so is the kernel K_R of R.

This implies the above theorem, because $\|R\|_{L^2\to L^2} \leq \underbrace{\iint |K_R(x,y)|^2 dx dy}_{\text{Hilbert-Schmidt norm}} < \infty$

<u>Proof</u>: Construct R of the form $R = P^*P + B^*B - M^2$ with B of order 0, properly supported.

Symbol of $P^*P - M^2$ is $|p|^2 - M^2$ up to terms of order -1

Therefore $M^2 - |p(x,\xi)|^2 > 0$ for $|\xi| > \overline{R}$, $\overline{R}$ suff. large.

Let $\chi \in C_c^\infty$, $\chi(\xi) = 1$ for $|\xi| \leq \overline{R} \implies b(x,\xi) = \chi(\xi)\,(M^2 - |p(x,\xi)|^2)^{1/2} \in S^0_{1,0}$

$\implies \exists$ properly supported B_0 w/ symbol b_0 s.t. $B_0^*B_0 = M^2 - P^*P + \underbrace{\text{operator of order } -1}_{E}$

Ansatz: $B = B_0 + B_1$ (B_1 order -1) $\implies M^2 - P^*P - B^*B$

$$= \underbrace{(M^2 - P^*P - B_0^*B_0)}_{=E} - B_0^*B_1 - B_1^*B_0 - \overbrace{B_1^*B_1}^{\text{order } -2}$$

$\overset{!}{=}$ order -2

leading symbol: $\quad 2b_1(x,\xi)\,b_0(x,\xi) = $ symbol of E

b_0 invertible for $|\xi| > \bar{R} \Rightarrow \chi(\xi)\,b_0^{-1}(x,\xi) \in S^0_{1,0}$

$\Rightarrow \exists B_1 \in S^{-1}_{1,0}$ properly supported w/ symbol b_1 + lower order terms.

$\Rightarrow \quad M^2 - P^*P - (B_0+B_1)^*(B_0+B_1)$ properly supported, order -2

Induction: B_j w/ symbol $b_j(x,\xi)$

Let $\quad b(x,\xi) \sim \sum_{j=0}^{\infty} b_j(x,\xi)$, B properly supp. w/ symbol b.

$\Longrightarrow R = P^*P + B^*B - M^2$ does the job $\quad \boxtimes$

Thm: (compactness)

Let $P = op(p) \in Op(S^0_{1,0}(\mathbb{R}^n))$ s.t. its kernel has compact support.

Assume $\overline{\lim}_{|\xi|\to\infty} |p(x,\xi)| < M$. $\Rightarrow \exists A_1 \in Op(S^0_{1,0})$ s.t.

- $A - A_1$ negligible, compactly supported kernel
- $\|A_1\| \le M$.

Therefore, if $\sup_x |p(x,\xi)| \xrightarrow{|\xi|\to\infty} 0 \Rightarrow A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ compact

Cor: Compactly supported operators of order <0 are compact on $L^2(\mathbb{R}^n)$.

Proof: Let $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi(x) \ge 0$, $\int \chi = 1$, $0 \le \hat{\chi} \le 1$.

(Existence: Let $\chi_0 \in C_c^\infty$, $\chi_0 \ge 0$, $\int \chi_0 = 1 \Rightarrow |\hat{\chi}_0| \le 1$. $\chi(x) := \int \chi_0(x+y)\,\chi_0(y)\,dy$ does the job, because $\hat{\chi} = |\hat{\chi}_0|^2$.)

$\chi_\varepsilon(x) := \varepsilon^{-n}\chi(x/\varepsilon)$, $P_\varepsilon u := Pu - P(\chi_\varepsilon * u)$

above $\Rightarrow \quad \|P_\varepsilon u\|^2_{L^2} \le M^2 \|u - \chi_\varepsilon * u\|^2_{L^2} + (R(u - \chi_\varepsilon * u), u - \chi_\varepsilon * u)$

$$\|u - \chi_\varepsilon * u\|_{L^2} = \|\mathcal{F}(u - \chi_\varepsilon * u)\|_{L^2} = \|\hat{u}(\xi) - \hat{\chi}(\varepsilon\xi)\hat{u}(\xi)\|_{L^2} \le \|\hat{u}\|_{L^2} = \|u\|_{L^2}$$

$R_\varepsilon u := R(u - \chi_\varepsilon * u)$, kernel $\quad K_R(x,y) - \int K_R(x,z)\,\varepsilon^{-n}\chi\left(\frac{z-y}{\varepsilon}\right)dz$

$$= K_R(x,y) - \int K_R(x, y+\varepsilon z)\,\chi(z)\,dz$$

$\Rightarrow$ supp K_{R_ε} bounded indep. of $\varepsilon \le 1$, $\sup_{x,y} |K_{R_\varepsilon}(x,y)| \xrightarrow{\varepsilon\to 0^+} 0$

Thm: Let $p \in S^d_{1,0} \Rightarrow \exists\, q \in S^{-d}_{1,0}$ s.t. $p \# q \sim q \# p \sim 1$

$\Longleftrightarrow$ p elliptic.

q is called a "parametrix symbol" and is unique up to $S^{-\infty}_{1,0}$.
The operator op(q) can be chosen to be properly supported.

Proof: „$\Rightarrow$" simple: Use that $p \# q = pq +$ lower order terms $= 1$.

„$\Leftarrow$" Let $\chi \in C_c^{\infty}$, $\chi(\xi) = 0$ for $|\xi| \geq 2R$.

Set $q_{-d}(x,\xi) := p(x,\xi)^{-1}(1-\chi(\xi))$

$$\Rightarrow \quad p \# q_{-d} \sim 1 - r \quad , r \in S^{-1}_{(1,0)}$$
$$q_{-d} \# p \sim 1 - s \quad s \in S^{-1}_{(1,0)}$$

$$\Rightarrow \quad \underbrace{p \# q_{-d}}_{\sim 1-r} \# (1 + r + r^{\#2} + \dots + r^{\# M}) \sim 1 - r^{\# M+1}$$

Let $r' \in S^{-1}_{(1,0)}$, $r' \sim \sum_{j=1}^{\infty} r^{\# j}$. $\Rightarrow \quad p \# \underbrace{q_{-d} \# r'}_{=:q} \sim 1$

Similarly $\exists \tilde{q} : \tilde{q} \# p \sim 1$. But

$$q = 1 \# q \sim (\tilde{q} \# p) \# q = \tilde{q} \# (p \# q) \sim \tilde{q} \# 1 = \tilde{q} \Rightarrow q \sim \tilde{q}$$

$$\Rightarrow \quad q \# p = \underbrace{(q - \tilde{q})}_{\sim 0} \# p + \underbrace{\tilde{q} \# p}_{\sim 1} \sim 0 \# p + 1 \sim 1.$$

Conclusion: $q := q_{-d} \# \left(1 + \sum_{j \geq 1} (1 - p \# q_{-d})^j\right)$

does the job. $\boxed{\text{ﾂ}}$

Remark: The construction is "microlocal" (see DifFun 2-notes):

One can construct a parametrix "in a point $x_0 \in \Omega$ and in a direction ξ_0" if only $|p(x,\xi)| \geq C|\xi|^d$ for x near x_0 and $\frac{\xi}{\|\xi\|}$ near $\frac{\xi_0}{\|\xi_0\|}$.

Classical expansion of the parametrix

One can directly construct the homogeneous terms in the expansion of the parametrix, if $op(p)$ is elliptic, $p \sim \sum_{\ell=0}^{\infty} p_{d-\ell}$, $p_{d-\ell}(x, t\xi) = t^{d-\ell} p(x,\xi)$ $\forall x\ \forall |\xi| \geq 1\ \forall t \geq 1$.

We first show that, as claimed previously, if p_d is invertible for $|\xi| \geq 1$, then $q_{-d} := (1-\chi(\xi))\, p_d^{-1} \in S^{-d}$ (χ some suitable cut-off fct, $\chi(\xi) = 0\ \forall |\xi| \geq 1/2$)

In fact, $|p_d^{-1}| \leq C(x)\ \forall |\xi| = 1$ by assumption $\Rightarrow |p_d^{-1}(x,\xi)| = |\xi|^{-d} |p_d^{-1}(x, \frac{\xi}{|\xi|})| \leq C(x) \langle\xi\rangle^{-d}$

Note that $p_d\, p_d^{-1} = 1 \Rightarrow (D_x p_d)\, p_d^{-1} + p_d\, D_x p_d^{-1} = 0$

$$\Rightarrow \forall |\xi| \geq 1: \ |D_x p_d^{-1}| \leq |p_d^{-1}|\,|D_x p_d|\,|p_d^{-1}| \leq C(x)^2 \langle\xi\rangle^{-2d}\, C_{01}(x) \langle\xi\rangle^{d} = \tilde{C}_0(x) \langle\xi\rangle^{d}$$

Similarly $\forall |\xi| \geq 1: \ |D_\xi p_d^{-1}| \leq |p_d^{-1}|\,|D_\xi p_d|\,|p_d^{-1}| \leq C(x)^2 \langle\xi\rangle^{-2d}\, C_{10}(x) \langle\xi\rangle^{d-1} = \tilde{C}_{10}(x) \langle\xi\rangle^{d-1}$

Using Leibniz' rule we obtain by induction:

$$|D_x^\beta D_\xi^\alpha p_d^{-1}| \leq \tilde{C}_{\alpha\beta}(x) \langle\xi\rangle^{d-|\alpha|} \quad \forall |\xi| \geq 1$$

or $\quad q_{-d} = (1-\chi)\, p_d^{-1} \in S^{-d}$.

Now we look for a parametrix symbol of the form $q \sim \sum_{\ell=0}^{\infty} q_{-d-\ell}$ s.t. $op(q) \circ op(p) \sim op\left(\sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha q\, \partial_x^\alpha p\right) \sim 1$. Collecting terms of the same order, we obtain

$$\sum_{m=0}^{\infty} \sum_{\substack{\alpha, \ell, k \\ m = k+|\alpha|+\ell}} \underbrace{\frac{1}{\alpha!} \underbrace{D_\xi^\alpha q_{-d-k}}_{\text{order } -d-k-|\alpha|} \underbrace{\partial_x^\alpha p_{d-\ell}}_{\text{order } d-\ell}}_{\text{order } -k-|\alpha|-\ell} \overset{!}{=} 1$$

or $$q_{-d-m} = -\left(\sum_{\substack{\alpha, \ell, k \\ m = k+|\alpha|+\ell \\ \alpha \neq 0}} \frac{1}{\alpha!} D_\xi^\alpha q_{-d-k}\, \partial_x^\alpha p_{d-\ell} \right) q_{-d} \quad (m \geq 1).$$

By definition, $q_{-d-m} \in S^{-d-m}$, $q_{-d-m}(x, t\xi) = t^{-d-m} q(x,\xi)$ $\forall |\xi| \geq 1$ $\forall t \geq 1$

and, if $q \sim \sum q_{-d-m} \in S^{-d}$, then $op(q) \circ op(p) \sim 1$.

Therefore, q is the (unique up to $S^{-\infty}$) parametrix symbol.

Corollary: a) Let $P \in Op(S^d_{1,0})$ properly supported, $u \in \mathcal{D}'$ and $Pu = f \in H^s_{loc} \implies u \in H^{s+d}_{loc}$

b) Let $P \in Op(S^d_{1,0})$, $u \in \mathcal{E}'$ and $Pu = f \in H^s_{loc}$ $\implies u \in H^{s+d}_{loc}$

Proof: Let Q be a properly supported parametrix, i.e.

$$R := Q \circ P - 1 \in S^{-\infty}_{1,0}(\Omega \times \Omega, \mathbb{R}^n)$$

$$\implies u = Q \circ Pu + Ru = Qf + Ru$$

$R : \mathcal{E}' \to C^\infty \implies Ru \in C^\infty$

$Q : H^s_{loc} \to H^{s+d}_{loc} \implies Qf \in H^{s+d}_{loc}$ $\implies u = Qf + Ru \in H^{s+d}_{loc} \implies$ b)

If P is properly supported $\implies R = Q \circ P - 1$ properly supported as composition / sum of properly supported operators

$\implies R : \mathcal{D}' \to C^\infty \implies$ a) $\square$

Elliptic estimates: Let p elliptic, $P = op(p) \implies \forall u \in W^{s,p}_{comp}(\Omega)$

$$\|u\|_{W^{s,p}} = \|Q \circ Pu + Ru\|_{W^{s,p}} \leq C(\|Pu\|_{W^{s-d,p}} + \|u\|_{s-1})$$

since R is of order $s-1$.

Uniqueness: Let $Pu = Pu'$ $\implies u - u' = (QP + R)(u-u') = R(u-u') \in C^\infty$.

So any two solutions of $Pu = f$ differ by a smooth function.

Singular support: For $u \in \mathcal{D}'(\Omega)$ define $\Omega_\infty(u) = \bigcup \{\omega \subset \Omega \text{ open} : u|_\omega \in C^\infty(\omega)\}$

sing supp $u := \Omega \setminus \Omega_\infty(u)$ closed subset of supp u.

For differential operators supp $Pu \subset$ supp u, they are "local".

Pseudo differential operators are "pseudo local".

Prop. 7.11: $\quad$ sing supp $Pu \subset$ sing supp u

See Gerd's book for a direct proof.
We show more generally:

Prop: Let K be a generalized integral operator with kernel $\in \mathcal{D}'(\Omega\times\Omega)$ s.t. $K: C_c^\infty \to C^\infty$, $\mathcal{E}' \to \mathcal{D}'$ and the kernel $|_{\Omega\times\Omega\setminus \mathrm{diag}(\Omega)} \in C^\infty$

$\Longrightarrow$ sing supp $Ku \subset$ sing supp u.

Proof: Let $x \notin$ sing supp u $\Rightarrow \exists\, \varphi \in C_c^\infty : \varphi \equiv 1$ near x, $\varphi u \in C_c^\infty \Rightarrow K(\varphi u) \in C^\infty$

On the other hand $K(u-\varphi u) \in \mathcal{D}'(\Omega)$ is given by

$\langle K(x,y), u(y) - \varphi(y) u(y) \rangle$. By the continuity of $\langle \cdot,\cdot \rangle$,

this can be differentiated w.r.t. x: $\partial_x^\alpha \langle K, \cdots \rangle = \langle \partial_x^\alpha K, \cdots \rangle$. $\blacksquare$

Prop: If P is elliptic $\Rightarrow$ sing supp Pu = sing supp u if $u \in \mathcal{E}'$.
If P is also properly supported, it holds for $u \in \mathcal{D}'$.

Proof: We only need to show „$\supseteq$".

Let Q be a properly supported parametrix, $QP - 1 = R$

$\Rightarrow$ sing supp u = sing supp $(QPu - \underbrace{Ru}_{\in C^\infty})$

$=$ sing supp $QPu \underset{7.11}{\subset}$ sing supp Pu. $\blacksquare$

Gårding's inequality

If p is a matrix-valued symbol, we let $\mathrm{Re}\, p := \frac{p+p^*}{2}$.

A $k\times k$-matrix-valued symbol p of order d is said to be <u>strongly elliptic</u> if $\langle v, (\mathrm{Re}\, p) v \rangle \geq \alpha(x) |\xi|^d \|v\|^2 \quad \forall |\xi| \geq R$, R large, $\forall v \in \mathbb{C}^k$

Recall from DifFun 2 that $[H^s, H^t]_\theta = H^r$, $r = s + \theta(t-s)$,

so that $\|u\|_{H^r} \leq \varepsilon \|u\|_{H^s} + C(\varepsilon) \|u\|_{H^t}$ for $\varepsilon \in (0,1]$.

See also Gerd's book for an elementary approach (7.22)

Thm 7.23: (Gårding) Let P properly supp. strongly elliptic of order $d>0$ on Ω_1, let $\Omega \subset \Omega_1$ open s.t. $\overline{\Omega} \subset\subset \Omega_1$.

$$\Longrightarrow \quad \exists c_0>0 \ \exists k \in \mathbb{R} : (\operatorname{Re} Pu, u) \geq c_0 \|u\|^2_{H^{d/2}} - k \|u\|^2_{L^2}$$

Proof: (only scalar case, matrix case is similar).

Let $p^0(x,\xi) := \sqrt{\operatorname{Re} p}$ for $|\xi| \geq R$ and extend p^0 smoothly to all ξ. $\Longrightarrow$ p^0 elliptic symbol of order $\frac{d}{2}$.

Let $P^0 := \operatorname{op} p^0 \Longrightarrow \operatorname{Re} \operatorname{op} p = P^{0*}P^0 + S$, S of order $d-1$

Let Λ_s denote a properly supported operator with symbol $\langle \xi \rangle^s$.

$$\Longrightarrow \quad S = \underbrace{(S \Lambda_{-\frac{d}{2}})}_{S_1} \underbrace{\Lambda_{\frac{d}{2}}}_{S_2} + \underset{\uparrow}{R_1}$$

order $-\infty$, because $\Lambda_{-\frac{d}{2}} \Lambda_{\frac{d}{2}} \sim 1$

S_1, S_2, R_1 can all be chosen to be properly supported

Let Q a properly supported parametrix for P^0: $QP^0 + R_2 = \mathbb{1}$.

Properly supported $\Longrightarrow$ $\exists$ bounded Ω', Ω'' s.t.

$$\begin{array}{ccccccccccccc}
\Omega & \subset & \overline{\Omega} & \subset & \Omega' & \subset & \overline{\Omega'} & \subset & \Omega'' & \subset & \overline{\Omega''} & \subset & \Omega_1 \\
\cup & & & & \cup & & & & \cup & & & & \\
\operatorname{supp} u & & & & \begin{array}{c}\operatorname{supp} Pu \\ \operatorname{supp} P^0 u \\ \operatorname{supp} S_1 u \\ \operatorname{supp} S_1^* u \\ \operatorname{supp} S_2 u\end{array} & & & & \begin{array}{c}\operatorname{supp} QPu \\ \operatorname{supp} S_1 S_2 u \\ \operatorname{supp} R_1 u \\ \operatorname{supp} R_2 u\end{array} & & & &
\end{array}$$

Now
$$\operatorname{Re}(Pu,u) = (\operatorname{Re} Pu, u) = (P^{0*}P^0 u, u) + (S_1 S_2 u, u) + (R_1 u, u)$$
$$= \|P^0 u\|^2_{L^2} + (S_2 u, S_1^* u) + (R_1 u, u)$$

As previously:
$$\|u\|^2_{H^{d/2}} \leq \left(\|QP^0 u\|_{H^{d/2}} + \|R_2 u\|_{H^{d/2}} \right)^2$$
$$\leq c\left(\|P^0 u\|_{L^2} + \|u\|_{L^2}\right)^2 \leq 2c\left(\|P^0 u\|^2_{L^2} + \|u\|^2_{L^2}\right)$$
$$\Longrightarrow \|P^0 u\|_{L^2} \geq \frac{1}{2c} \|u\|^2_{H^{d/2}} - \|u\|^2_{L^2}$$

• $|(S_2 u, S_1^* u)| \le \|S_2 u\|_{L^2} \|S_1^* u\|_{L^2} \le C' \|u\|_{H^{d/2}} \|u\|_{H^{d/2-1}}$

$$\le \frac{C'}{2}\left(\varepsilon \|u\|^2_{H^{d/2}} + \frac{1}{\varepsilon}\|u\|^2_{H^{d/2-1}}\right)$$

If $\frac{d}{2}-1>0$, by the remark at the beginning of the section

$$\|u\|_{H^{\frac{d}{2}-1}} \le \varepsilon' \|u\|^2_{H^{d/2}} + C(\varepsilon')\|u\|^2_{L^2}$$

$$\le (4C)^{-1}\|u\|^2_{H^{d/2}} + C''\|u\|^2_{L^2}$$

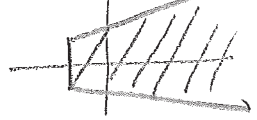
• $|(R_1 u, u)| \le \|R_1 u\|_{L^2}\|u\|_{L^2} \le C\|u\|^2_{L^2}$

$\Longrightarrow$ $\operatorname{Re}(P u, u) \ge (4C)^{-1}\|u\|^2_{H^{d/2}} - C'''\|u\|^2_{L^2}$ $\square$

Corollary: (Thm. 7.24) $\Omega \subset \mathbb{R}^n$ bounded, open, A strongly elliptic differential operator of order d on a neighborhood of $\overline{\Omega}$.

$\Longrightarrow$ $d = 2m$, the Dirichlet realization A_γ of A on $L^2(\Omega)$ with domain $D(A_\gamma) = H_0^m(\Omega) \cap D(A_{max})$ is variational, and $D(A_\gamma) \subseteq H^{2m}_{loc}(\Omega)$ (Cor. 7.20)

the spectrum and numerical range of A_γ are contained in a sector of the form



It is more difficult to see that $D(A_\gamma) \subseteq H^{2m}(\Omega)$ if $\partial\Omega$ smooth.

Special classes of operators

Hilbert-Schmidt operators: An operator A on a Hilbert space H is Hilbert-Schmidt provided that for some (then: every) orthonormal basis $\{e_n\}_{n\in I}$ of H: $\|A\|_{HS}^2 := \sum_n \|Ae_n\|_H^2 < \infty$.

If A is an integral operator, $Au(x) = \int_{\mathbb{R}^n} K(x,y)\, u(y)\, dy$ for some measurable function $K: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, we have the following characterization:

$$A \text{ Hilbert-Schmidt} \iff K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$$

Then $\|A\|_{HS}^2 = \int dx \int dy\, |K(x,y)|^2$.

We only show „$\Leftarrow$": Let $\{e_n\}_n$ an orthonormal basis of $L^2(\mathbb{R}^n)$

$$\Rightarrow \sum_n \|Ae_n\|^2 = \sum_n \int dx \left| \int dy\, K(x,y)\, e_n(y) \right|^2$$

$$= \int dx \sum_n \left| \langle K(x,\cdot), \overline{e_n(\cdot)} \rangle_{L^2(\mathbb{R}^n_y)} \right|^2$$

$$\overset{\text{(Parseval)}}{=} \int dx\, \| K(x,\cdot) \|^2_{L^2(\mathbb{R}^n)} = \int dx \int dy\, |K(x,y)|^2 < \infty$$

Proposition: Let $a \in S^{-\frac{n}{2}-\varepsilon}_{1,0}(\mathbb{R}^n, \mathbb{R}^n)$, for some $\varepsilon > 0$, compactly supported in x.
$\Rightarrow op(a)$ Hilbert-Schmidt.

Proof: $K(x,y) = \mathcal{F}^{-1}_{\xi \to x-y}\, a(x,\xi)$ as L^2-Fourier transform for any given x.

$$\Rightarrow \iint |K|^2 = \iint \left| \mathcal{F}^{-1}_{\xi \to x-y}\, a(x,\xi) \right|^2 dx\, dy$$

$$\overset{(z = x-y)}{=} \iint \left| \mathcal{F}^{-1}_{\xi \to z}\, a(x,\xi) \right|^2 dx\, dz$$

$$\overset{\text{(Plancherel)}}{=} \iint |a(x,\xi)|^2\, dx\, d\xi < \infty. \quad \boxtimes$$

For a Hilbert-Schmidt operator A, the singular values $\{s_j\}_{j=1}^{\infty}$ (eigenvalues of A^*A) are $\in \ell^2(\mathbb{N})$.

Trace class operators: If $\{s_j\} \in \ell^1(\mathbb{N})$, A is said to be trace class.
They have a well-defined trace, $\operatorname{tr} A := \sum_n \langle Ae_n, e_n \rangle$.
More generally, one defines the Schatten classes

$$S_p := \{ A \in \mathbb{K}(H) : \underbrace{\sum_j |s_j|^p}_{=: \|A\|_{S_p}^p} < \infty \}. \qquad (\|A\|_\infty := \sup_j s_j)$$

Properties:
- $(S_p, \|\cdot\|_{S_p})$ is a Banach space
- Hölder inequality $\|AB\|_{S_1} \leq \|A\|_{S_p} \|B\|_{S_{p'}}$
  $\frac{1}{p} + \frac{1}{p'} = 1$
- $$S_p \xrightarrow{\sim} S_{p'}^* \qquad , \frac{1}{p} + \frac{1}{p'} = 1, \ 1 \leq p < \infty,$$
  $$A \longmapsto (B \mapsto \operatorname{Tr}(AB))$$
- S_1 = trace class operators
  S_2 = Hilbert-Schmidt operators
- Lidskii: $\operatorname{tr} A = \sum$ (Eigenvalues of A) $\quad \forall A \in S_1$.

One also considers weak Schatten classes

$$\Sigma_p := \{ A \in S_\infty : \sup_j j^{1/p} s_j < \infty \}.$$

The Hölder inequality in particular implies that the product of Hilbert-Schmidt operators is trace class.

Proposition: Let $a \in S_{1,0}^{-n-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$, for some $\varepsilon > 0$, with compactly supported kernel $\Longrightarrow$ $\operatorname{op}(a) \in S_1$ and

$$\operatorname{Tr} \operatorname{op}(a) = (2\pi)^n \int dx \int d\xi \, a(x, \xi).$$

In Julie's presentation about Lidskii's theorem and the DifFun2 lecture, we have sketched that: $\operatorname{Tr} A = \int_{\mathbb{R}^n} K(x,x)\,dx$, where $K(x,y) = \mathcal{F}^{-1}_{\xi \to x-y}\, a(x,\xi)$ as (continuous) Fourier transform of an L^1-function $\Longrightarrow$ $K(x,x) = (2\pi)^{-n}\int d\xi\, a(x,\xi)$ and $\operatorname{Tr} A = (2\pi)^{-n}\int dx \int d\xi\, a(x,\xi)$. See also the discussion in Shubin II, Section 13.2.

To show that actually $op(a) \in S_1$, we are going to write $op(a)$ as a product of Hilbert-Schmidt operators.

Note that, as the kernel of $op(a)$ has compact support, for any $\chi \in C_c^\infty(\mathbb{R}^n)$, $\equiv 1$ on a large ball, we have

$$op(a) = op(a)\chi = op(a(x,\xi)\,[\xi]^{t}\,[\xi]^{-t})\chi = \underbrace{op(a(x,\xi)\,[\xi]^{t})}_{\in S^{-n-\varepsilon+t}_{1,0}} \circ op([\xi]^{-t})\chi$$

The kernel of $op([\xi]^{-t})\chi$ is $\underbrace{\left(\mathcal{F}^{-1}_{\xi\to x-y}[\xi]^{-t}\right)}_{=: F(x-y)} \chi(y)$

$$\Longrightarrow \int dx \int dy\, |F(\underbrace{x-y}_{=z})|^2 |\chi(y)|^2 = \int dz \int dy\, |F(z)|^2 |\chi(y)|^2 = (2\pi)^{-n}\int d\xi \int dy\, [\xi]^{-2t} |\chi(y)|^2 < \infty \text{ if } t > \tfrac{n}{2}$$

$t := \frac{n+\varepsilon}{2}$ $\Longrightarrow$ $op(a)$ is a product of Hilbert-Schmidt operators $\square$

More generally, one can show that if $a \in S_{1,0}^{-\frac{n}{p}-\varepsilon}$ and the kernel of $op(a)$ is compactly supported (or a if a has merely compact support in x) $\Longrightarrow$ $op(a) \in S_p$.

Fredholm operators:

Exercise: If $a \in S^d_{1,0}(\mathbb{R}^n, \mathbb{R}^n)$ is globally estimated (i.e. the constant $C_{\alpha\beta K}$ in the definition of $S^d_{1,0}$ can be chosen independent of K) and if a^{-1} exists and is bounded outside a compact subset C of $\mathbb{R}^n \times \mathbb{R}^n$, then $\exists\, q \in S^{-d}_{1,0}(\mathbb{R}^n, \mathbb{R}^n)$ s.t. $op(q) \circ op(a) - 1$ and $op(a) \circ op(q) - 1$ are compact.

Hint: Construct a parametrix starting with $q_0 = \chi(x,\xi)\, a^{-1}(x,\xi)$ where $\chi \equiv 0$ on a neighborhood of C.
"Use that the kernel of a globally estimated operator is $\leq C\,|x-y|^{-N}$ $\forall N$ $\forall\, |x-y|>1$ to write the remainders $op(q) \circ op(a) - 1$, $op(a) \circ op(q) - 1$ in the form $op(c+r)$, where $c \in S^{-1}_{1,0}$ has compactly supported kernel and the kernel of $r \in S^{-\infty}$ is $\in L^2(\mathbb{R}^n \times \mathbb{R}^n)$.
See Hörmander, The Analysis of Linear Partial Differential Operators III, Thm 19.3.1."

An operator $A \in \mathcal{L}(H_1, H_2)$ is said to be <u>Fredholm</u> if $\exists\, Q \in \mathcal{L}(H_2, H_1)$ s.t. $AQ - \mathbb{1}$ and $QA - \mathbb{1}$ are compact.

Thomas will say more about them next week.

Coordinate changes

Let $\Omega, \underline{\Omega} \subseteq \mathbb{R}^n$ open, $K: \Omega \to \underline{\Omega}$ smooth diffeomorphism

pull-back: $K^*: C^\infty(\underline{\Omega}) \xrightarrow{\sim} C^\infty(\Omega)$ continuous by chain rule and Leibniz rule
$\varphi \longmapsto \varphi \circ K$

K^* preserves test fcts.: $K^*: C_c^\infty(\underline{\Omega}) \xrightarrow{\sim} C_c^\infty(\Omega)$

Transformation law for integrals: $\int_\Omega (K^*\varphi)\psi = \int_{\underline{\Omega}} \varphi(\psi \circ K^{-1}) |\det DK^{-1}|$
$(\varphi, \psi \in C_c^\infty)$

So we can extend K^* to distributions as

$$\langle K^* u, \varphi \rangle := u\left((K^{-1})^* \varphi \, |\det DK^{-1}|\right), \quad u \in \mathcal{D}'(\underline{\Omega}), \ \varphi \in C_c^\infty(\Omega)$$

$\rightsquigarrow K^*: \mathcal{D}'(\underline{\Omega}) \xrightarrow{\sim} \mathcal{D}'(\Omega)$

K^* restricts to $L_{p,loc}(\underline{\Omega}) \to L_{p,loc}(\Omega)$, $C^m(\underline{\Omega}) \to C^m(\Omega)$.
If (e.g.) K extends to a diffeomorphism of $\mathbb{R}^n$ which is linear outside some compact set, the chain rule shows that $K^*: H^k(\underline{\Omega}) \xrightarrow{\sim} H^k(\Omega)$ $\forall k \in \mathbb{N}$ continuously.
As the adjoint $(K^*)^* = |\det DK^{-1}| \cdot (K^{-1})^*$ also $H^k(\Omega) \to H^k(\underline{\Omega})$
we get $K^*: H^k(\underline{\Omega}) \xrightarrow{\sim} H^k(\Omega)$ $\forall k \in \mathbb{Z}$.

The Riesz-Thorin theorem from DifFun 2 then implies
$K^*: H^s(\underline{\Omega}) \xrightarrow{\sim} H^s(\Omega)$ $\forall s \in \mathbb{R}$ continuously.

Operators: Let $p \in S^m_{1,0}(\Omega \times \Omega, \mathbb{R}^n)$, $P = op(p(x,y,\xi)): C_c^\infty(\Omega) \to C^\infty(\Omega)$

Define $\underline{P} := (K^{-1})^* P K^*: C_c^\infty(\underline{\Omega}) \to C^\infty(\underline{\Omega})$
$\underline{u} \longmapsto (P(\underline{u} \circ K)) \circ K^{-1}$

Thm 8.1: • $\underline{P}$ is a pseudodifferential operator on $\underline{\Omega}$

• $\underline{P} \sim \sum_{j,k} op\left(p(K^{-1}(x), K^{-1}(y), \psi_{jk}(x,y)\xi)\, \mathcal{D}_{jk}(x,y)\right)$,
$\psi_{jk}, \mathcal{D}_{jk}$ defined in the proof

- If $p \in S^m_{1,0}(\Omega, \mathbb{R}^n)$, $\underline{P} \sim op(\underline{p}(x,\xi))$ where

$$\underline{p}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D^\alpha_\xi \, p(\kappa^{-1}(x), {}^t(D\kappa)\xi) \, \varphi_\alpha(\kappa^{-1}(x), \xi)$$

and $\varphi_\alpha \in S^{|\alpha|/2}(\Omega, \mathbb{R}^n)$ is some polynomial in ξ of degree $\leq \frac{|\alpha|}{2}$,
$\varphi_0 = 1$, $\varphi_\alpha = 0$ for $|\alpha| = 1$.

<u>Corollary</u>: a) If $p \in S^m(\Omega, \mathbb{R}^n)$, $\underline{p} \in S^m(\underline{\Omega}, \mathbb{R}^n)$ with principal symbol

$$p_m(\kappa^{-1}(x), {}^t(D\kappa)\xi).$$

b) If $p \in S^m_{1,0}(\Omega, \mathbb{R}^n)$ is elliptic, $\underline{p}$ is elliptic.

<u>Proof</u>: The corollary follows directly from the asymptotic expansion.

<u>Proof of Thm</u>: Writing down what $\underline{P}$ is:

$$\underline{P}\,\underline{u}(x) = \lim_{\varepsilon \to 0^+} \int_\Omega dy \int_{\mathbb{R}^n} d\xi \; e^{i(\kappa^{-1}(x) - y)\xi} \chi(\varepsilon\xi) \, p(\kappa^{-1}(x), y, \xi) \, u(\kappa(y))$$

$$\overset{(y = \kappa^{-1}(z))}{=} \lim_{\varepsilon \to 0^+} \int_{\underline{\Omega}} dz \int_{\mathbb{R}^n} d\xi \; e^{i(\kappa^{-1}(x) - \kappa^{-1}(z))\xi} \chi(\varepsilon\xi) \, p(\kappa^{-1}(x), \kappa^{-1}(z), \xi)$$

$$\cdot \left| \det D\kappa^{-1}(z) \right| u(z) \qquad (*)$$

Define ϕ s.t. $(\kappa^{-1}(x) - \kappa^{-1}(z))\xi = \sum_{k,j} \phi^{kj}(x,z)\,(x_k - z_k)\,\xi_j$.

By the inverse fct thm, ϕ is well-defined, smooth and invertible for $(x,z) \in \underline{K} \times \underline{K}$, $K \subset\subset \underline{\Omega}$ if $|x - z| < \varepsilon$ for some small $\varepsilon > 0$ depending on $\underline{K}$. Let $\psi(x,z) := \phi(x,z)^{-1}$, $0 \leq \tilde{\chi} \in C_c^\infty(\underline{K} \times \underline{K})$, $\tilde{\chi}(x,z) \equiv 1$ for $|x - z| < \frac{1}{3}$ and $\equiv 0$ for $|x - z| > \frac{2}{3}$.

$D(x,z) := \left| (\det D\kappa^{-1}(z)) (\det \psi(x,z)) \tilde{\chi}(x,z) \right|$. Assume that

$p(\kappa^{-1}(x), \kappa^{-1}(z), \cdot) = 0$ for x or $z \notin K$ $\Rightarrow$

$$\underline{P}\,\underline{u}(x) \underset{{}^t\phi(x,z)\xi =: \eta}{=} \lim_{\varepsilon \to 0^+} \int_{\underline{\Omega}} dz \int_{\mathbb{R}^n} d\eta \; e^{i(x-z)\eta} \chi(\varepsilon \psi(x,z)\xi)$$

$$\cdot \, p(\kappa^{-1}(x), \kappa^{-1}(z), \psi(x,z)\eta) \, D(x,z) \, u(z)$$

$$+ \lim_{\varepsilon \to 0^+} \int_{\underline{\Omega}} dz \int_{\mathbb{R}^n} d\xi \; (\ldots \text{ as in } (*)) \cdot (1 - \tilde{\chi})$$

Because of the $(1-\tilde{\chi})$-factor, the integrand of the second term vanishes near $x=z$. Integrating by parts in ξ we find that the corresponding kernel is smooth. Thus the second term is negligible. (This also follows from Lemma 7.9)
Concerning the first term, we have noted before that the definition of $\mathrm{op}(p)$ does not depend on the cut-off fct. χ as long as $\chi(0)=1$. So we can define $\mathrm{op}(\underline{p})$ using $\chi\circ\psi$, so that

$$\underline{P} = \mathrm{op}\big(\overbrace{p(\kappa^{-1}(x),\kappa^{-1}(y),\psi(x,y)\xi)\,\mathcal{D}(x,y)}^{\underline{p}}\big) + \text{negligible}$$

The chain and product rules of differentiation show that the $\underline{p} \in S^m_{1,0}(\underline{\Omega}\times\underline{\Omega},\mathbb{R}^n)$. If p is classical, so is $\underline{p}$.

If $p(\kappa^{-1}(x),\kappa^{-1}(z),\cdot)\neq 0$ for some x or $z\notin K$, we choose a locally finite partition of 1 on Ω, $1=\sum_{j=0}^{\infty}\varphi_j$,

$$p=\sum_{j,k}\underbrace{\varphi_j(x)\,p(x,y,\xi)\,\varphi_k(y)}_{=:\,p_{jk}}, \quad P=\sum_{j,k}\varphi_j P\varphi_k = \sum \mathrm{op}(p_{jk})$$

Then $\underline{P} = \sum_{j,k}\underline{\varphi_j}\,P\,\underline{\varphi_k} = \sum_{j,k}\mathrm{op}(\underline{p}_{jk})$, $\underline{\varphi_j}(x)=\varphi_j(\kappa^{-1}(x))$, and the above discussion applies to those p_{jk} with $\operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k\neq\emptyset$.
If $\operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k=\emptyset \Rightarrow \mathrm{op}(\underline{p}_{jk})$ negligible.

Finally, the formula for the asymptotic expansion can be deduced by taking left-symbols, $\mathrm{op}(\underline{p}(x,y,\xi))\sim\mathrm{op}(\underline{p}_1(x,\xi))$ as in Thm 7.13 $\boxtimes$

Manifolds:

Def: X n-dimensional smooth manifold $:\Longleftrightarrow$

- X Hausdorff top. space, 2^{nd} countable
- X locally diffeomorphic to $\mathbb{R}^n$

The latter means that there are an open cover $X = \bigcup_k U_k$

open $V_k \subset \mathbb{R}^n$ and homeomorphisms $\kappa_k : U_k \to V_k$

s.t. $\forall j,k: \kappa_j \circ \kappa_k^{-1} : \kappa_k(U_k \cap U_j) \to \kappa_j(U_k \cap U_j)$ diffeo. $(*)$

A family of such κ_k is called an atlas.
An atlas is said to be maximal if it contains any homeomorphism κ_k satisfying $(*)$ for all κ_j in the atlas. We will only consider manifolds endowed with a maximal atlas.
On a compact manifold, a finite atlas suffices, and we may assume the V_j to be bounded and disjoint.

$u \in C^\infty(X), C^m(X), L_{p,loc}(X), H^s_{loc}(X), W^{s,p}_{loc}(X) :\Longleftrightarrow u \circ \kappa_j^{-1} \in C^\infty(V_j), C^m(V_j), L_{p,loc}(V_j), H^s_{loc}(V_j), W^{s,p}_{loc}(V_j) \quad \forall j$

$u \in C_c^\infty(X) \Longleftrightarrow u \in C^\infty(X)$, supp u compact

This is independent of the choice of κ_j, because $u \circ \tilde{\kappa}_j^{-1} = (u \circ \kappa_j^{-1}) \circ \underbrace{(\kappa_j \circ \tilde{\kappa}_j^{-1})}_{\text{smooth}}$.

Key Lemma 8.4. X cpct manifold.

1°: For any finite open cover $\{U_1, \dots, U_J\}$ of X exists an associated partition of 1, i.e. $\{\varphi_j\}_{j=1}^J \subseteq C_c^\infty(U_j)$, $\varphi_j \geq 0$, $\sum \varphi_j = 1$.

Generalizing matrices of operators acting on k-tuples of functions, we consider operators acting on sections of a vector bundle over a manifold.

Let X be a manifold.

<u>Examples:</u> • The trivial N-dim vectorbundle over X is $X \times \mathbb{C}^N$.

<u>projection onto "base"</u> X: $\pi : X \times \mathbb{C}^N \to X$
$(x, v) \mapsto x$

<u>fibre</u> over $x \in X$: $\pi^{-1}(\{x\}) = \mathbb{C}^N$

<u>zero section</u>: $X \times \{0\} \subseteq X \times \mathbb{C}^N$

<u>sections</u>: $f : X \to \mathbb{C}^N$ (i.e. maps f s.t. $\pi \circ f = id_X$)

• tangent bundle = space of tangent vectors to a manifold.

<u>Def:</u> A <u>smooth vector bundle</u> E over X is a manifold E with • a surjection $\pi : E \to X$ $\in C^\infty$

• $\pi^{-1}(\{x\}) \cong \mathbb{C}^N$ $\forall x \in X$

• local diffeomorphisms $E \to X \times \mathbb{C}^N$, i.e.
$\forall x \in X$ $\exists$ open neighborhood U $\exists$ $\Psi : \pi^{-1}(U) \to V \times \mathbb{C}^N$ diffeomorphism, $\Psi(\pi^{-1}(\{x\})) = \{K(x)\} \times \mathbb{C}^N$ $\forall x \in U$ for some coord. fct. $K : U \to V$ of X and
$\pi^{-1}(\{x\}) \xrightarrow{\Psi} \{K(x)\} \times \mathbb{C}^N \to \mathbb{C}^N$ <u>linear</u> isomorphism.

(Ψ, U, V): <u>local trivialization</u>

Ψ_1, Ψ_2 local trivializations $\rightsquigarrow g_{12} := \Psi_1 \circ \Psi_2^{-1} : K_2(U_1 \cap U_2) \times \mathbb{C}^N$
$\downarrow$
$K_1(U_1 \cap U_2) \times \mathbb{C}^N$

smooth family of invertible $N \times N$ matrices, "<u>transition function</u>"

A family of such $\{g_{ij}\}_{i,j \in I}$ satisfying $g_{ij} g_{ji} = id$, $g_{ij} g_{jk} g_{ki} = id$ defines a unique vector bundle: $E \cong I \times X \times \mathbb{C}^N / \sim$ $(i, x, v) \sim (i', x, g_{i'i} v)$.

<u>section</u>: $f : X \to E$ s.t. $\pi \circ f = id_X$

In each local trivialization, a section is a fct $X \to \mathbb{C}^N$.

$\Rightarrow$ Can define $C^\infty, C_c^\infty, C^m, L^p_{loc}, H^s_{loc}, W^{s,p}_{loc}$ as for manifolds.

Distribution densities on X:

Def: A distribution density u on X is a collection of distributions

$u_K \in \mathcal{D}'(V_K)$, K coordinate fcts, s.t. $\forall\, K = K_j K_k^{-1}$

$$\langle u_{K_k} \circ K^{-1}, \varphi \circ K^{-1} \rangle = \langle |\det DK|\, u_{K_j}, \varphi \rangle$$

Let X be compact.
Then, with the notation from Lemma 8.4.1°, the action of u on $\varphi \in C^\infty(X)$,

$$\langle u, \varphi \rangle_X = \sum_{j=1}^{J} \langle u_{K_j}, (\psi_j \varphi) \circ K_j^{-1} \rangle_{V_{K_j}}$$

is independent of the choice of coordinates.
Distribution densities are sections of the 1-dim. bundle which is trivial on each chart U_{K_j} and has transition functions $|\det D(K_j K_k^{-1})|$.

So under a change of coordinates, a distribution density is multiplied by $|\det DK|$, which means that they do not quite generalize smooth functions. However, fixing some positive distribution density a, we get

$$C^\infty(X) \hookrightarrow C_c^\infty(X)', \quad f \mapsto \langle f, \varphi \rangle := \int_X f \varphi a$$

: Unlike for functions, the integral of a distributional density does not depend on the choice of coordinates. There are natural L^p-spaces associated to a choice of a.
There is often a canonical choice of a, e.g. given by a Riemannian metric on the manifold. We shall assume from now on that such a choice has been made and identify the space of distributional densities with $C_c^\infty(X)'$.

Def.: X compact, $H^s(X) := \{ u \in \mathcal{D}'(X) : \forall j \ \underbrace{(\psi_j u) \circ \kappa_j^{-1}}_{\text{extend by } 0} \in H^s(\mathbb{R}^n) \}$

Norm: $\|u\|^2_{H^s} := \sum_{j=1}^{J} \|(\psi_j u) \circ \kappa_j^{-1}\|^2_{H^s(\mathbb{R}^n)}$

Basic Properties:

- $H^s(X)$ Hilbert space
- $C^\infty(X)$ dense in $H^s(X)$: Approximate $(\psi_j u) \circ \kappa_j^{-1}$ by $v_j^\varepsilon \in C_c^\infty(V_j)$, then $u^\varepsilon := \sum_j v_j^\varepsilon \circ \kappa_j \xrightarrow{\varepsilon \to 0^+} u$.
- dual space $(H^{-s})' = H^s$ (can choose norms appropriately)

Pseudodifferential operators on X:

Def.: $P : C^\infty(X) \to C^\infty(X)$ pseudo diff. operator on X of order d if $P_j = \kappa_j^{-1 *} P \kappa_j^* : C_c^\infty(V_j) \to C^\infty(V_j)$, $v \mapsto P(v \circ \kappa_j) \circ \kappa_j^{-1}$ is a pseudodiff. operator on V_j of order d $\forall j$.

Notions of classical or elliptic operators are similarly defined. The principal symbol is invariantly defined as a function on the cotangent bundle T^*X.

Lemma 8.4.2°: $\exists$ local coordinates $\kappa_i : U_i \to V_i$, $i = 1, \dots, I$, and $\exists$ subordinate partition of 1 $\{\rho_1, \dots, \rho_J\}$ s.t. any four fcts $\rho_j, \rho_k, \rho_\ell, \rho_m$ are supported in some U_i.

(Similar for vector bundles.)

Products: $PQ = \sum_{j,k,\ell,m} \underbrace{\rho_j P \rho_k \rho_\ell Q \rho_m}_{\text{supported in some } U_i \xrightarrow{\kappa_i} V_i \subseteq \mathbb{R}^n}$

Composition rules on $\mathbb{R}^n \Rightarrow$

- PQ ΨDO on X.
- symbol computed by usual rules in local coordinates.

Similar: Adjoint ΨDO on X, principal symbol of adjoint = adjoint of principal symbol.

Thm 8.5: P ΨDO on X of order d $\Rightarrow$ $P : H^s(X) \to H^{s-d}(X)$ cont. $\forall s$.

Proof: ρ_k as is 8.4.2°, $P = \sum \rho_j P \rho_k$ — supported in U_i for some i. The symbol of

$K_i^{-1*} \rho_j P \rho_k K_i^*$ in (x,y)-form is compactly supported in $V_i \times V_i$.

$\Rightarrow$ extends (by 0) to $\in S^d_{1,0}(\mathbb{R}^{2n}, \mathbb{R}^n)$ $\Rightarrow$ defines cont. operator $H^s(\mathbb{R}^n) \to H^{s-d}(\mathbb{R}^n)$. Assertion follows from definition of $H^s(X)$. ☐

Remark: Similarly: If $\sup_x |p(x,\xi)| \xrightarrow{|\xi|\to\infty} 0$ for some choice of local coordinates $\Rightarrow P : H^s(X) \to H^s(X)$ compact.

Thm 8.6 / 8.11: P elliptic of order d $\Leftrightarrow$ $\exists$ elliptic Q of order $-d$ s.t.

$$PQ = \mathbb{1} + R_1, \quad QP = \mathbb{1} + R_2, \quad R_1, R_2 \text{ order } -\infty.$$

Proof: Let ψ_j as in 8.4.1°, $\zeta_j, \Theta_j \in C_c^\infty(U_j)$ s.t. $\zeta_j \equiv 1$ in a neighborhood of supp ψ_j, $\Theta_j \equiv 0$ in a neighborhood of supp ζ_j. P_j elliptic on V_j $\Rightarrow$ $\exists$ parametrix Q_j' on V_j.

$$C_j' := \underbrace{(\Theta_j \circ K_j^{-1})}_{\underline{\Theta}_j} P_j (\Theta_j \circ K_j^{-1}) \underbrace{(\zeta_j \circ K_j^{-1})}_{\underline{\zeta}_j} Q_j' \underbrace{(\psi_j \circ K_j^{-1})}_{\underline{\psi}_j} - (\psi_j \circ K_j^{-1})$$

$$= (\underline{\Theta}_j P_j \underline{\zeta}_j Q_j' - \mathbb{1}) \underline{\psi}_j = (\underline{\Theta}_j \underbrace{P_j Q_j'}_{\sim 1} - \mathbb{1}) \underline{\psi}_j + \underline{\Theta}_j P_j (\underline{\zeta}_j - 1) Q_j' \underline{\psi}_j$$

order $-\infty$ $\Leftarrow$ disjoint support

$$\sim \underbrace{(\underline{\Theta}_j - 1)}_{=0 \text{ on supp } \underline{\psi}_j} \underline{\psi}_j = 0$$

$\Rightarrow$ C_j' negligible operator, kernel supported in supp $\underline{\Theta}_j \times$ supp $\underline{\psi}_j$

$\Rightarrow$ defines negligible operator C_j on X.

Let $Qu = \sum_j \left(\zeta_j Q_j'((\psi_j u) \circ K_j^{-1})\right) \circ K_j$ $\Rightarrow$

$$(PQ - \mathbb{1}) u = \sum_j P[\underline{\zeta}_j Q_j'((\psi_j u) \circ K_j^{-1}) \circ K_j] - \psi_j u$$

$$= \sum_j \underbrace{(\Theta_j P \Theta_j [\zeta_j Q_j'((\psi_j u) \circ K_j^{-1}) \circ K_j] - \psi_j u)}_{C_j} + Ru \sim 0.$$

R negligible, because $1 - \Theta_j$, ζ_j have disjoint supports.

Left-parametrix similar, Q is both-sided parametrix ☐

Thm 8.11:
- Elliptic operators are Fredholm as operators $H^s(X) \to H^{s-d}(X)$.
- Their kernel is contained in $C^\infty(X)$ and does not depend on $s \in \mathbb{R}$.
- Their index is independent of $s \in \mathbb{R}$. If $(P_t)_{t \in [0,1]}$ is a family of elliptic operators depending continuously on t $\Rightarrow$ $\operatorname{ind} P_t = \text{constant}$.

Proof: As operators R of negative order satisfy $\sup_x |r(x,\xi)| \xrightarrow{|\xi| \to \infty} 0$, they are compact on $H^s(X)$, hence also from $H^s(X) \to H^{s-d}(X)$. Theorem 8.6 and Atkinson's theorem (Lemma 8.9) give the first assertion.

The second assertion follows from elliptic regularity, Cor. 7.20.

For the third assertion, if P is elliptic, so is P^*, hence its kernel is independent of $s \in \mathbb{R}$ $\Rightarrow$

$$\operatorname{index} P = \dim \ker P - \dim \ker P^* \quad \text{independent of } s \in \mathbb{R}$$

The last statement is the homotopy invariance of the index. $\square$

Remark: In particular, the index only depends on the principal symbol. If P_0, P_1 have the same principal symbol, $P_t = t P_0 + (1-t) P_1$ shows $\operatorname{ind} P_0 = \operatorname{ind} P_1$.

Lemma (I.7.1 in Shubin): Let $s \in \mathbb{R}$ $\Rightarrow$ $\exists$ classical elliptic ΨDO Λ_s of order s with positive principal symbol.

Proof: If $M \subset \mathbb{R}^n$ open, $\Lambda_s \sim op([\xi]^s)$ as before.

If M is a manifold, using notation as in Thm. 8.6, let Λ_s^j an operator with these properties in V_j.

Define $\Lambda_s u := \sum_j \left(\zeta_j \Lambda_s^j\left((\psi_j u) \circ \kappa_j^{-1} \right)\right) \circ \kappa_j$. $\square$

More on Sobolev spaces:

We give two further descriptions of the topology of $H^s(X)$:

Prop I.7.3 (in Shubin):

The topology on $H^s(X)$ is equivalently defined by
- the norm $\|u\|'_{H^s(X)} := \|\Lambda_s u\|^2_{L^2(X)} + \sum_{k=1}^N \|Q_k R_s u\|^2_{L^2(X)}$

where $R_s = \Lambda_{-s} \Lambda_s - \mathbb{1}$ (order $-\infty$) and $Q_1, \dots, Q_N$ are differential operators on X s.t. their linear combinations with $C^\infty(X)$-coefficients generate all differential operators of order $d \geq s$.

or • The seminorms $\|u\|_A := \|Au\|_{L^2(X)}$, A ΨDO of order s.

<u>Proof:</u> Both $\|\cdot\|_{H^s}$, $\|\cdot\|'_{H^s}$ define Banach space structures on $H^s(X)$.
By the closed graph theorem, all such norms are equivalent.

The seminorms $\|\cdot\|_A$ define a stronger topology, as

$$\|u\|'_{H^s} \leq \|u\|_{\Lambda_s} + \sum_{k=1}^{N} \|u\|_{Q_k R_s}.$$

To show the converse we need to show $\|Au\|_{L^2} \leq C\|u\|'_{H^s}$
if A ΨDO of order s. $u = \Lambda_s \Lambda_{-s} u + R_s u$

$$\Rightarrow \|Au\|_{L^2} \leq \|\underbrace{(A\Lambda_{-s})}_{\text{order } 0}\Lambda_s u\|_{L^2} + \|\underbrace{AR_s}_{\text{order } -\infty} u\|_{L^2}$$

$$\underset{(L^2\text{-boundedness})}{\leq} C\left(\|\Lambda_s u\|_{L^2} + \|u\|_{L^2}\right) \leq 2C\|u\|'_{H^s} \quad \boxed{}$$

<u>Cor:</u> The topology on $H^s(X)$ is independent of the various choices made in the definition of $\|\cdot\|_{H^s}$ and $\|\cdot\|'_{H^s}$.

<u>Cor:</u> (Rellich's theorem)

$H^s(X) \hookrightarrow H^{s'}(X)$ compact if $s > s'$.

<u>Proof:</u>
$$\Lambda_{s'} u = (\Lambda_{s'}\Lambda_{-s})(\Lambda_s u) - (\Lambda_{s'} R) u$$
$$= \underbrace{(\Lambda_{s'}\Lambda_{-s})}_{\substack{\text{order} < 0 \\ \Rightarrow \text{compact}}}(\Lambda_s u) - (\Lambda_{s'}\Lambda_{-s})(\Lambda_s R_s u) + \underbrace{(\Lambda_{s'} R_s)}_{\text{compact}}(R_s u)$$

If $B \subseteq H^s(X)$ bounded $\Rightarrow$ $\Lambda_{s'} B \subseteq$ Compact operators (bounded subsets of $L^2(X)$)
$\subseteq$ Compact subset of $L^2(X)$

$\Rightarrow \Lambda_{s'} B$ precompact in $L^2(X)$

Similar: $Q R_{s'} B$ precompact in $L^2(X)$ for any diff. operator Q

Definition of $\|u\|'_{H^{s'}} = \|\Lambda_{s'} u\|_{L^2} + \sum_j \|Q_j R_{s'} u\|_{L^2}$

$\Rightarrow$ B precompact in $H^{s'}(X)$. $\boxed{}$

<u>Cor:</u> A operator of order d $\Rightarrow$ $A: H^s(X) \to H^{s-d-\varepsilon}(X)$ compact $\forall \varepsilon > 0$.

Proof: $H^s(X) \xrightarrow[\text{bounded}]{A} H^{s-d}(X) \xrightarrow[\text{compact}]{} H^{s-d-\varepsilon}(X)$ compact $\square$

Thm 7.6: (Sobolev embedding)

Let $s > \frac{n}{2} + k \Rightarrow H^s(X) \hookrightarrow C^k(X)$ is compact

Proof: Since differential operators of order k are bounded $H^s \to H^{s-k}$, we may assume $k=0$. We only need to show that

$H^s(X) \hookrightarrow C^k(X)$ bounded $\forall s > \frac{n}{2}+k$, since then

$H^s(X) \underset{\text{compact}}{\hookrightarrow} H^{s-\varepsilon}(X) \underset{\text{bounded}}{\hookrightarrow} C^k(X)$ is compact by Thm. 7.4.

It is enough to show $H^s(K) \hookrightarrow C^k(K)$ for $K \subset U_i$ contained in one of the coordinate patches. But there we know it from the Sobolev embedding theorem on $\mathbb{R}^n$, Thm 6.11 in Grubb's book. $\square$

Cor 7.4: $\bigcap_s H^s(X) = C^\infty(X)$, $\bigcup_s H^s(X) = \mathcal{D}'(X)$.

One may further refine theorem 8.5 and Rellich's theorem using our above results for trace-class, Hilbert-Schmidt and Schatten class operators on $\mathbb{R}^n$. The very same arguments lead to:

Cor: Let A be a pseudo diff. operator on X of order $d \leq 0$.

Then $A: H^s \to H^s$ belongs to

- the p-th Schatten class $S_p(H^s)$ if $d < -\frac{\dim X}{p}$
- the p-th weak Schatten class $\Sigma_p(H^s)$ if $d \leq -\frac{\dim X}{p}$

Applications to the spectrum

Let A be an elliptic ΨDO of order d on a compact manifold X, $d>0$.
Consider the unbounded operator A_0 on $L^2(X)$ defined by $D(A_0) = H^d(X)$
$A_0 u = Au \quad \forall u \in D(A_0)$.

Prop 8.41 A_0 is a closed operator.

Proof: We have to show that if $u_n \in H^d$ s.t. $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} Au_n = f$

exist in L^2 $\Rightarrow u \in H^d$ and $Au = f$.

But convergence in $L^2 \Rightarrow$ convergence in $\mathcal{D}'$

A cont. in $\mathcal{D}'$ $\Rightarrow f = \lim_{n\to\infty} Au_n = A \lim_{n\to\infty} u_n = Au$, so $Au = f$.

$u \in H^d$ by elliptic regularity $\square$

Cor: A_0 is the maximal realization of A, i.e. $D(A_0) = \{u \in L^2 : Au \in L^2\}$
It coincides with the minimal realization: $A_0 = \overline{A|_{C^\infty}}^{L^2}$.

Proof: The first assertion follows from elliptic regularity / Prop 8.4.
The second is obtained by noting $\overline{C^\infty(X)}^{H^d} = H^d$. $\square$

Thm 8.2: Let $\lambda \notin \sigma(A_0) \Rightarrow (A_0 - \lambda)^{-1}$ is (the restriction from $\mathcal{D}'(X)$ of) an elliptic pseudodiff. operator of order $-m$ and hence compact.

Proof: Wlog $\lambda = 0$. $\quad AB = 1 - R$, R smooth kernel $R(x,y)$

$$\Rightarrow A^{-1} = B + A^{-1}R$$

B pseudodiff. of order $-d$. Show $A^{-1}R$ negligible.

As $A^{-1}: H^s \to H^{s+d}$ $\forall s$ continuously $\Rightarrow A^{-1}: \underset{\bigcap H^s}{\overset{}{C^\infty}} \to \underset{\bigcap_s H^{s+d}}{C^\infty}$ cont.

$\Rightarrow A^{-1}R$ has smooth kernel $R_1(x,y) := [A^{-1} R(\cdot,y)](x)$ $\square$

__Remark:__ In general, if $(\lambda_0 - A_0)^{-1}$ is compact for some λ_0

$\Rightarrow$ $(\lambda - A_0)^{-1} = (\lambda_0 - A_0)^{-1}\left(1 + (\lambda - \lambda_0)(\lambda - A_0)^{-1}\right)$ shows that $(\lambda - A_0)^{-1}$ is compact $\forall \lambda \notin \sigma(A_0)$.

__Thm 8.3/8.4:__ Let $d > 0$ $\Rightarrow$
- $\sigma(A_0) = \mathbb{C}$ or
- $\sigma(A_0) = \{\text{eigenvalues } \lambda_j \text{ with at most } \infty \text{ as limit points}\}$

If A_0 is self-adjoint, $\mathbb{R} \ni |\lambda_j| \nearrow \infty$ and there exists an orthonormal basis of $L^2(X)$ consisting of eigenfunctions.

__Proof:__ Let $\sigma(A_0) \neq \mathbb{C}$, $\lambda_0 \notin \sigma(A_0)$. Thm 8.2 $\Rightarrow$ $(A_0 - \lambda_0)^{-1} =: R_{\lambda_0}$ compact $\Rightarrow$ $\sigma(R_{\lambda_0}) = \{\text{eigenvalues } r_j \text{ with only limit point } 0\}$

As $(A_0 - \lambda_0)^{-1}$ injective $\Rightarrow$ $r_j \neq 0$.

$$R_{\lambda_0}\varphi_j = r_j \varphi_j \iff (A_0 - \lambda_0)\varphi_j = r_j^{-1}\varphi_j$$
$$\iff A_0 \varphi_j = \underbrace{(\lambda_0 + r_j^{-1})}_{\lambda_j}\varphi_j$$

$\Rightarrow$ $\{\lambda_j\}$ discrete set

$$A_0 - \lambda = (A_0 - \lambda_0)\left(\mathbb{1} - (\lambda - \lambda_0)(A_0 - \lambda_0)^{-1}\right), \text{ so } \lambda \in \sigma(A_0)$$

$\iff$ $\lambda \neq \lambda_0$ and $\frac{1}{\lambda - \lambda_0} \in \sigma((A_0 - \lambda_0)^{-1})$

$\Rightarrow$ $\sigma(A_0) = \{\lambda_j\}$

If A_0 is self-adjoint $\Rightarrow$ R_{λ_0} selfadjoint, compact

$\Rightarrow$ $\exists$ orthonormal basis of eigenfcts of R_{λ_0} of $L^2(X)$.

By the above, these are eigenfcts of A_0 $\square$

Prop: Let A be a classical pseudodifferential operator of order d on a compact manifold s.t. $A: H^{s+d}(X,E) \to H^s(X,F)$ is Fredholm for some $s \in \mathbb{R}$. Then A is elliptic.

Sketch of proof: Wlog $d=0$ (Otherwise consider $A\Lambda^{-d}$) and $s=0$. Let Q be a Fredholm inverse of A, $QA = 1+K$, K compact.

$\Rightarrow (*)$ $\|u\|_{L^2(X,E)} = \|QAu - Ku\|_{L^2(X,E)} \leq C\left(\|Au\|_{L^2(X,F)} + \|Ku\|_{L^2(X,E)}\right) \quad \forall u \in L^2(X,E)$

Since A^* is Fredholm, too, there is a similar estimate for A^*.

We show that $(*)$ implies the injectivity of a_d. The same considerations applied to A^* lead to the injectivity of the principal symbol of A^*, which is a_d^*, i.e. the surjectivity of a_d.

Let $a_d(x,\xi)$ be the expression of the principal symbol in some local coordinates around a point (x_0,ξ_0), $|\xi_0|=1$. Let w be a section of E supported in this coordinate neighborhood. Then the family of operators $R_\lambda w(x) = \lambda^{-n/2} e^{i\langle x, \lambda \xi_0\rangle} w((x-x_0)\lambda^{-1/2})$, $\lambda > 0$

satisfies
- $\|R_\lambda w\|_{L^2(X,E)} = \|w\|_{L^2(X,E)}$
- $R_\lambda w \xrightarrow{\lambda\to 0^+} 0$ (weak convergence).

$\Rightarrow$
- $\|KR_\lambda w\|_{L^2(X,E)} \xrightarrow{\lambda\to 0^+} 0 \quad \xrightarrow{(*)} \quad \|w\|_{L^2} \leq C\,\|a_d(x_0,\xi_0)w(x_0)\|$
- $\|AR_\lambda w\|_{L^2(X,F)} \xrightarrow{\lambda\to 0^+} \|a_d(x_0,\xi_0)w(x_0)\|$

Choosing w s.t. $w(x_0) \neq 0$ we obtain the injectivity of a_d $\boxtimes$

One can extend this proof to nonclassical symbols.

Properties of pseudodiff. operators on a compact manifold X

Functional analysis	S^k	$S^k_{1,0}$	pseudodiff. operator
bounded $H^s \to H^{s-d}$	$\Longleftrightarrow$	$\Longleftarrow$	order d
Fredholm $H^s \to H^{s-d}$	$\Longleftrightarrow$	$\Longleftrightarrow$	elliptic of order d
bounded $\mathcal{D}' \to C^\infty$	$\Longleftrightarrow$	$\Longleftrightarrow$	order $-\infty$ $\Longleftrightarrow$ smooth kernel
compact $H^s \to H^s$	$\Longleftrightarrow$	$\Longleftrightarrow$	$\overline{\lim}_{\lvert\xi\rvert\to\infty}$ (symbol) $= 0$
compact $H^s \to H^s$	$\Longleftrightarrow$	$\Longleftarrow$	order < 0
Hilbert–Schmidt : $H^s \to H^s$	$\Longleftrightarrow$	$\Longleftrightarrow$	$\int dx \int d\xi \, (\text{symbol})^2 < \infty$
Hilbert–Schmidt : $H^s \to H^s$	$\Longleftrightarrow$	$\Longleftarrow$	order $< -\frac{\dim X}{2}$
trace class $H^s \to H^s$	$\Longleftrightarrow$	$\Longleftarrow$	order $< -\dim X$
p-th Schatten class $S_p(H^s)$	$\Longleftrightarrow$	$\Longleftarrow$	order $< -\frac{\dim X}{p}$
p-th weak Schatten class $\Sigma_p(H^s)$	$\Longleftrightarrow$	$\Longleftarrow$	order $\leq -\frac{\dim X}{p}$