

Simple analytical index formulas

Let $A: H_1 \rightarrow H_2$ Fredholm. Two fundamental observations:

Prop: a) $\ker A^*A = \ker A$, $\ker AA^* = \ker A^*$.

b) Let $E_\lambda := \{u \in H_1 : A^*Au = \lambda u\}$, $F_\lambda := \{u \in H_2 : AA^*u = \lambda u\}$.

Then $\dim E_\lambda = \dim F_\lambda$.

Proof: b) $(AA^*)AE_\lambda = A(A^*A)E_\lambda = \lambda AE_\lambda$, so $AE_\lambda \subseteq F_\lambda$

$(A^*A)A^*F_\lambda = A^*(AA^*)F_\lambda = \lambda A^*F_\lambda$, so $A^*F_\lambda \subseteq E_\lambda$

As $E_\lambda = (\lambda 1_{H_1})E_\lambda = A^*AE_\lambda \subseteq A^*F_\lambda \Rightarrow A^*F_\lambda = E_\lambda$
 $F_\lambda = (\lambda 1_{H_2})F_\lambda = AA^*F_\lambda \subseteq AE_\lambda \Rightarrow AE_\lambda = F_\lambda \Rightarrow E_\lambda \cong F_\lambda$.

a) " \supseteq " clear, " \subseteq ": $u \in \ker A^*A \Rightarrow 0 = \langle u, A^*Au \rangle = \|Au\|^2 \Rightarrow u \in \ker A$
 $u \in \ker AA^* \Rightarrow 0 = \langle u, AA^*u \rangle = \|A^*u\|^2 \Rightarrow u \in \ker A^*$

□

Cor: Let A be an elliptic (pseudo-) differential operator on a compact manifold X . Then

$$\text{ind}(A) = \zeta(1 + A^*A, z) - \zeta(1 + AA^*, z) \quad \forall z$$

$$= \text{tr } e^{-tA^*A} - \text{tr } e^{-tAA^*} \quad \forall t > 0$$

$$= a_{\dim X}(A^*A) - a_{\dim X}(AA^*) \quad \text{where } \text{tr } e^{-tQ} = \sum_j a_j(Q) t^{-\frac{j - \dim X}{2 \dim X}}$$

Proof: Let $\lambda_j = j$ -th eigenvalue of A^*A , $\mu_j = j$ -th eigenvalue of AA^*

$$\Rightarrow \zeta(1 + A^*A, z) - \zeta(1 + AA^*, z) = \sum_j (1 + \lambda_j)^{-z} - \sum_j (1 + \mu_j)^{-z}$$

$$\stackrel{(\text{Prop})}{=} \sum_{j: \lambda_j=0} 1 - \sum_{j: \mu_j=0} 1 = \dim \ker A - \dim \ker A^*$$

$$= \text{ind } A. \quad \text{for } \text{Re } z > \frac{\dim X}{2 \dim A}$$

$$\text{Similarly, } \text{tr } e^{-tA^*A} - \text{tr } e^{-tAA^*} = \sum_j e^{-t\lambda_j} - \sum_j e^{-t\mu_j}$$

$$\stackrel{(\text{Prop})}{=} \sum_{j: \lambda_j=0} e^{-t \cdot 0} - \sum_{j: \mu_j=0} e^{-t \cdot 0} = \text{ind}(A)$$

The asymptotic expansion of the left-hand side as $t \rightarrow 0^+$ is

$$\sum_j (a_j(A^*A) - a_j(AA^*)) t^{-\frac{j - \dim X}{2 \dim A}} \stackrel{!}{=} \text{ind}(A) \quad \text{indep. of } t$$

$$\Rightarrow a_j(A^*A) = a_j(AA^*) \quad \forall j \neq \dim X \text{ and } a_{\dim X}(A^*A) - a_{\dim X}(AA^*) = \text{ind } A.$$

□

A slightly different type of trace formula for the index is the following:

Prop: Let $A: H_1 \rightarrow H_2$ Fredholm, Q such that $(1-QA)^N$ and $(1-AQ)^N$ are trace class for some $N \in \mathbb{N}$. Then

$$\text{ind}(A) = \text{tr}((1-QA)^N) - \text{tr}((1-AQ)^N).$$

Proof: In the proof of Atkinson's theorem we constructed Q_0 s.t.

$$1 - Q_0 A = P_{\ker A}, \quad \text{where } P_{\ker A}, P_{\text{coker } A} \text{ are the orthogonal}$$

$$1 - A Q_0 = P_{\text{coker } A}$$

projections onto $\ker A$ resp. $\text{coker } A$.

$$\Rightarrow \text{ind}(A) = \dim \ker A - \dim \text{coker } A = \text{tr } P_{\ker A} - \text{tr } P_{\text{coker } A}$$

$$= \text{tr}(1 - Q_0 A) - \text{tr}(1 - A Q_0)$$

Let $N=1$, $1-QA$, $1-AQ$ trace class. Note $\text{tr } ST = \text{tr } TS$ if S, T trace class \Rightarrow

$$\begin{aligned} \text{ind}(A) &= \text{tr}(1 - Q_0 A) - \text{tr}(1 - A Q_0) \\ &= \text{tr}(1 - QA - (Q_0 - Q)A) - \text{tr}(1 - AQ - A(Q_0 - Q)) \\ &= \text{tr}(1 - QA) - \text{tr}(1 - AQ) \\ &\quad - \underbrace{\text{tr}((Q_0 - Q)A) + \text{tr}(A(Q_0 - Q))}_{=0} \\ &= \text{tr}(1 - QA) - \text{tr}(1 - AQ) \end{aligned}$$

For $N > 1$; with $(1-QA)^N, (1-AQ)^N$ trace class, consider

$$Q' := Q(1 + (1-AQ) + \dots + (1-AQ)^{N-1})$$

$$\Rightarrow AQ' = (1 - (1-AQ))(1 + (1-AQ) + \dots + (1-AQ)^{N-1})$$

$$= 1 - (1-AQ)^N, \text{ so } 1 - AQ' = (1-AQ)^N \text{ trace class}$$

$$Q'A = Q(1 + (1-AQ) + \dots + (1-AQ)^{N-1})A$$

Since $Q(AQ)^k A = (QA)^{k+1}$, $Q'A = QA(1 + (1-QA) + \dots + (1-QA)^{N-1})$

$$= 1 - (1-QA)^N \in 1 + \text{trace class}$$

because $x(1+x+\dots+(1-x)^{N-1}) = 1 - (1-x)^N$

Therefore $\text{ind } A = \text{tr}(1 - Q'A) - \text{tr}(1 - AQ') = \text{tr}((1-QA)^N) - \text{tr}((1-AQ)^N)$ □